This exam consists of 2 sections. You do not have to show any work for the problems in **Part I**. Solutions to problems in **Part II**, however, must show work to earn full credit. For neither part are you allowed to use calculators. Pledge your exam before turning it in. You have 90 minutes for this exam.

Good luck!

Part I: No justification is necessary for the following questions.

1. What is the definition of a linear transformation?
2. Let V be the vector space of all polynomials $p(t)$ of degree less or equal than 4 such that $p(3) = 0$.
   (a) What is $\dim(V)$?
   (b) Write down a basis for V.
3. Let $v_1, \ldots, v_k$ be linearly independent vectors in a vector space V. Let w be the zero vector. Are $w, v_1, \ldots, v_k$ linearly independent?
4. Let $v_1, \ldots, v_k$ be linearly dependent vectors in a vector space V. Assume that $w \in \text{span}\{v_1, \ldots, v_k\}$. How many solutions does the equation $\lambda_1 v_1 + \cdots + \lambda_k v_k = w$ have, i.e. how many such λ_i's can we find? (possible answers are: none, exactly one, 23, infinitely many,...)
5. Let $\varphi : V \to W$ be a linear transformation which is onto. Assume that $\dim(V) = k$, $\dim(W) = l$. What is $\dim(\text{Ker}(\phi))$?
6. Let $B = \{v_1; v_2; v_3\}$ be an ordered basis for a vector space V. Assume that $w \in V$ is a vector such that $c_B(w) = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}^t$. Let C be the ordered basis $\{v_3; v_1 - v_2; v_2\}$. What is $c_C(w)$?

Part II on reverse

Part II:

1. Consider
$$v_1 := \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, v_2 := \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}, v_3 := \begin{pmatrix} 2 \\ 6 \\ 4 \\ 2 \end{pmatrix}.$$

   (a) Show that v_1, v_2, v_3 are linearly independent.

   (b) How many vectors do you have to add to v_1, v_2, v_3 to get a basis for $\mathbb{R}^4$?

   (c) Find vectors $w_1, \ldots, w_l$ (note that l is your answer from (b)) such that $v_1, v_2, v_3, w_1, \ldots, w_l$ form a basis for $\mathbb{R}^4$.

2. Let V be the vector space of all polynomials of degree at most three.

   (a) Determine whether
$$p_1(t) := 1 - t, p_2(t) := -1 + 2t + 3t^2 + t^3, p_3(t) := t^2 + 5t^3, p_4(t) = -1 + 3t + 7t^2 + 7t^3$$
   are linearly independent or not.

   (b) Find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that
$$\lambda_1 p_1(t) + \lambda_2 p_2(t) + \lambda_3 p_3(t) + \lambda_4 p_4(t) = -3 + 8t + 18t^2 + 20t^3.$$
   (you do NOT have to find ALL such λ's).

3. Let V be the vector space of all polynomials of degree at most three with ordered basis $B = \{1; t-1; t^2-1; t^3-1\}$. Furthermore let $W = \mathbb{R}^4$ with the ordered basis $\{2e_1; 3e_2; 4e_3; 5e_4\}$ (recall that $e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^t$ and so on). Now consider the linear transformation $\varphi : V \to W$ given by
$$\varphi(p(t)) = \begin{pmatrix} p(1) \\ p(2) \\ p''(1) \\ p''(2) - p(0) \end{pmatrix}$$

   (a) Find the matrix which represents φ with respect to the ordered bases B and C.

   (b) Is φ invertible? Justify your answer.

4. Let V be the vector space of all functions defined on $\mathbb{R}$. Define $\varphi : V \to V$ by $\varphi(f(x)) := f(x-2)$.

   (a) Show that φ is a linear transformation.

   (b) What is the inverse map, i.e. what is the linear transformation $\psi : V \to V$ such that $\psi(\varphi(f(x)) = f(x)$ and $\varphi(\psi(f(x)) = f(x)$ for all functions $f(x)$?