Solutions for Exam #2

Part I:

1. Since the dimensions of the column space and row space of a matrix are always the same, the answers in (a) and (b) will be equal. Further, the answer is given by the rank of A. Since A is already in row-echelon form, we can find the rank by counting leading columns. The answer in both parts is 2.

2. A **linear transformation** is a map between vector spaces, $T\colon V \to W$ which satisfies
$$\begin{aligned} T(v+v') &= T(v) + T(v'), \\ T(\alpha v) &= \alpha T(v), \end{aligned}$$
for all vectors v, v' in V and all scalars $\alpha \in \mathbb{R}$.

3. The desired formula is given by the dimension theorem, which says
$$\dim V = \dim \text{kernel of } \varphi + \dim \text{image of } \varphi.$$

4. Yes, the map $\varphi = \varphi_1 + \varphi_2$ is always a linear transformation. To see this, we check:
$$\begin{aligned} \varphi(\alpha v + \beta w) &= \varphi_1(\alpha v + \beta w) + \varphi_2(\alpha v + \beta w) \\ &= \alpha\varphi_1(v) + \beta\varphi_1(w) + \alpha\varphi_2(v) + \beta\varphi_2(w) \\ &= \alpha\varphi_1(v) + \alpha\varphi_2(v) + \beta\varphi_1(v) + \beta\varphi_2(v) \\ &= \alpha\big(\varphi_1(v) + \varphi_2(v)\big) + \beta\big(\varphi_1(w) + \varphi_2(w)\big) \\ &= \alpha\varphi(v) + \beta\varphi(w). \end{aligned}$$

5. Since
$$\begin{aligned} v_1 &= 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 \\ 0 &= 0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 \\ v_3 - v_2 &= 0 \cdot v_1 + (-1) \cdot v_2 + 1 \cdot v_3, \end{aligned}$$
we have
$$c_B(v_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad c_B(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad c_B(v_3 - v_2) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

6. Yes, the set is necessarily a basis for V. We know that V has dimension k, so every basis for V has size k. We know $\{v_1, \cdots, v_k\}$ is linearly independent. If it were not a basis, then we could extend it to a basis that had more than k elements. But this is impossible since V has dimension k.

7. Since
$$\begin{aligned} v_1 + v_2 &= 1 \cdot v_1 + 1 \cdot v_2 \\ v_2 &= 0 \cdot v_1 + 1 \cdot v_2, \end{aligned}$$
the matrix A must be $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Part II:

1. To reduce a spanning set of columns to a basis, we form a matrix from the columns, perform Gaussian elimination, and keep the original columns that become leading columns. In this case, we take the 3×4 matrix

$$A = [v_1\ v_2\ v_3\ v_4] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & -2 & 1 \end{bmatrix}.$$

One possible result of Gaussian elimination is the matrix

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Since the first three columns of G are leading columns, this tells us that the set $\{v_1, v_2, v_3\}$ is a basis. **Note**: Since the result of Gaussian elimination is not unique, and because you can list the vectors in any order when you start, other answers are possible. In this case, any three of the four given vectors gives a basis (but this is not true in general).

2. To decide if the given polynomials in $\mathscr{P}^5$ are linearly independent, we have to decide if there are any non-trivial solutions in real a, b, c for

$$a(1-t) + b(1+2t^2) + c(t+2t^2) = 0.$$

Using the standard basis $\{1; t; t^2; t^3; t^4\}$, we are really asking whether the system $Ax = \mathbf{0}$ has any non-trivial solutions for the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solving this system, we see that there are non-trivial solutions (for example, $a = 1, b = -1, c = 1$). Hence, the set of vectors is not linearly independent.

3. To compute the matrix that represents φ, we just need to evaluate each basis vector under φ and write the answer in terms of the basis vectors again. Let $f(t) = 1, g(t) = t, h(t) = t^2$ be the basis vectors in B. Then:

$$\begin{aligned} \varphi(1) &= f(2) \cdot (t+2) = 1 \cdot (t+2) \\ &= 2 \cdot 1 + 1 \cdot t + 0 \cdot t^2, \\ \varphi(t) &= g(2) \cdot (t+2) = 2 \cdot (t+2) \\ &= 4 \cdot 1 + 2 \cdot t + 0 \cdot t^2, \\ \varphi(t^2) &= h(2) \cdot (t+2) = 4 \cdot (t+2) \\ &= 8 \cdot 1 + 4 \cdot t + 0 \cdot t^2. \end{aligned}$$

Hence, the matrix representing φ is

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. First, we show that φ is a linear transformation. For any $p(t), q(t)$ in V and any $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} \varphi\big(\alpha p(t) + \beta q(t)\big) &= \int_0^1 \alpha p(t) + \beta q(t)\ dt \\ &= \int_0^1 \alpha p(t)\ dt + \int_0^1 \beta q(t)\ dt \\ &= \alpha \int_0^1 p(t)\ dt + \beta \int_0^1 q(t)\ dt \\ &= \alpha\varphi\big(p(t)\big) + \beta\varphi\big(q(t)\big). \end{aligned}$$

Now, we need to show that φ is onto. That is, for any real number c, we have to find a polynomial $p(t)$ in V so that $\varphi\big(p(t)\big) = c$. Well, try $p(t) = c$:

$$\begin{aligned} \varphi\big(p(t)\big) &= \int_0^1 c\ dt \\ &= ct\Big|_0^1 \\ &= c - 0 = c. \end{aligned}$$

This shows that φ is onto.

Finally, we want to find the dimension of the kernel of φ. We know that V has dimension 28. Also, since φ is onto, we know that the image of φ is $\mathbb{R}$, which has dimension 1. So by the dimension theorem, the kernel of φ has dimension $28 - 1 = 27$.