This exam consists of two sections. In Part I, decide whether each statement is either true or false. No justification is necessary for your answer. In Part II, there are 9 questions. Show your work on each problem to earn full credit. Pledge your exam before turning it in. Good luck!

Part I: State whether each of the following statements is True or False. You do not need to justify your answer.

1. If matrices A and B are similar, then they have the same characteristic polynomial.
Solution: Yes. We did this in class.

2. The only matrix of rank 0 is the zero matrix.
Solution: Yes.

3. For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$, the inner product of $\mathbf{x}$ and $\mathbf{y}$ is greater than or equal to 0.
Solution: No. For example take $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The the inner product is -1.

4. Let A be a $p \times p$ matrix. Then $A\mathbf{x} = \mathbf{0}$ has a solution if and only if A is nonsingular.
Solution: There's always the solution $\mathbf{x} = 0$.

5. Every square matrix is similar to an upper triangular matrix.
Solution: Yes, e.g. we have the Jordan form.

6. Every linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^2$ must be onto.
Solution: No, the linear transformation could for example just be zero.

7. The kernel of a linear transformation $T: V \to W$ is always a subspace of V.
Solution: Yes.

8. Given two bases B, B' for a vector space V, the change of basis matrix between them is always invertible.
Solution: Yes.

9. The algebraic multiplicities of the eigenvalues of a $p \times p$ matrix always add up to p.
Solution: Yes.

10. The geometric multiplicities of the eigenvalues of a $p \times p$ matrix always add up to p.
Solution: No. This is only the case when the matrix is diagonalizable.

Part II: Remember to show all work in order to earn full credit.

2. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -2 \\ 5 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Do S and T span the same subspace of $\mathbb{R}^4$?
Solution: We didn't really do such a problem this year. One way how you can solve it is by checking that the columns of T lie in the span of S and vice versa.
If you want to check whether the columns of T lie in the span of S you have to write down the 4×5–matrix given by the columns of S and T (in this order!) and get it into row–echelon form. If the last two columns (which correspond to T) are non–leading columns, then they lie in the span of the columns of S.

3. Let A be a square matrix with $\det A = 0$. Show that 0 must be an eigenvalue of A.
Solution: $l = 0$ is an eigenvalue if there exists $v \neq 0$ such that $Av = \lambda v = 0v = 0$. But such a v exists if A non–singular, which is the case when $\det(A) = 0$.

5. **Without using the Jordan form algorithm**, decide whether each of the following matrices is diagonalizable. Give a **brief** explanation for your answer.

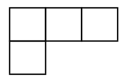
(a) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$
Solution: A is symmetric, hence diagonalizable.

(b) $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Solution: B is in Jordan form and not diagonal, hence B is not diagonalizable.

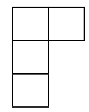
(c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$
Solution: C has three different eigenvalues, hence is diagonalizable.

6. A certain 8×8 matrix B has two distinct eigenvalues, $\lambda = 2$ and $\lambda = 4$. After performing the Jordan form algorithm, you find the following diagrams:

$\lambda = 2$



$\lambda = 4$



Write down a possible Jordan form for the matrix B.
Solution: The Jordan form will have one 2×2–block for $l = 2$ and two 1×1–blocks for $l = 2$ furthermore it will have one 3×3–block for $l = 4$ and one 1×1–block for $l = 4$.

8. Find values for $b, c \in \mathbb{C}$ that make the matrix

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & b \\ \frac{\sqrt{2}}{2}i & c \end{bmatrix}$$

unitary.
Solution: You have to find b, c such that $\frac{\sqrt{2}}{2}\bar{b} + \frac{\sqrt{2}}{2}i\bar{c} = 0$ and $|b|^2 + |c|^2 = 1$. The first equation tells you that $\bar{b} = -i\bar{c}$, i.e. $b = ic$. The second equation then tells you that $|b|^2 = \frac{1}{2}$. So we can take $c = \frac{\sqrt{2}}{2}$ and $b = ic$.

9. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2i \\ -1 \end{bmatrix} \in \mathbb{C}^3$.

   (a) Let W be the subspace of all vectors orthogonal to $\mathbf{v}$. Find a basis for W and give the dimension.

   Solution: We have to find all vectors $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ such that $w \cdot \begin{pmatrix} 1 \\ 2i \\ -1 \end{pmatrix} = 0$. But this just means that

$$w_1 \cdot 1 + w_2 \cdot \overline{2i} + w_3 \cdot (-1) = w_1 \cdot 1 + w_2 \cdot (-2i) + w_3 \cdot (-1) = 0.$$

   (note that we have to take complex conjugation of the second term!). So we have to solve the equation $w_1 \cdot 1 + w_2 \cdot (-2i) + w_3 \cdot (-1) = 0$. Which is equivalent to solving

$$\begin{pmatrix} 1 & -2i & -1 \end{pmatrix}$$

   Get it into row–echelon form (which in fact it is!) and read off the solutions from the non–leading columns. In particular the dimension is two.

(b) The following set of vectors is an ordered orthonormal basis for $\mathbb{C}^3$:

$$B = \left\{ \mathbf{u}_1 = \begin{bmatrix} \frac{1+i}{2} \\ 0 \\ \frac{1-i}{2} \end{bmatrix}; \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \mathbf{u}_3 = \begin{bmatrix} -\frac{i}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \right\}.$$

Express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.
Solution: We know that $v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + (v \cdot u_3)u_3$. Now we only have to be careful with computation of the dot–product, remember that we have to take complex conjugation for the second term.
Remark: I didn't teach the complex case this year, it's a little subtle because you can not reverse the order of v and u_i in the formula $v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + (v \cdot u_3)u_3$, i.e. we do NOT have $v = (u_1 \cdot v)u_1 + (u_2 \cdot v)u_2 + (u_3 \cdot v)u_3$.

(c) Find the coordinates $c_B(\mathbf{v})$.
Solution: That's your numbers from (b)!