Math 355 **Final Exam** **December 8, 2004**

This exam consists of two sections. In Part I, decide whether each statement is either true or false. No justification is necessary for your answer. In Part II, there are 9 questions. Show your work on each problem to earn full credit. Pledge your exam before turning it in. Good luck!

Part I: State whether each of the following statements is True or False. You do not need to justify your answer.

1. If matrices A and B are similar, then they have the same characteristic polynomial.

2. The only matrix of rank 0 is the zero matrix.

3. For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$, the inner product of $\mathbf{x}$ and $\mathbf{y}$ is greater than or equal to 0.

4. Let A be a $p \times p$ matrix. Then $A\mathbf{x} = \mathbf{0}$ has a solution if and only if A is nonsingular.

5. Every square matrix is similar to an upper triangular matrix.

6. Every linear transformation $T\colon \mathbb{R}^5 \to \mathbb{R}^2$ must be onto.

7. The kernel of a linear transformation $T\colon V \to W$ is always a subspace of V.

8. Given two bases B, B' for a vector space V, the change of basis matrix between them is always invertible.

9. The algebraic multiplicities of the eigenvalues of a $p \times p$ matrix always add up to p.

10. The geometric multiplicities of the eigenvalues of a $p \times p$ matrix always add up to p.

Part II: Remember to show all work in order to earn full credit.

1. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & 3 \\ 2 & 5 & -1 & 5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

Find all solutions to $A\mathbf{x} = \mathbf{b}$.

2. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -2 \\ 5 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Do S and T span the same subspace of $\mathbb{R}^4$?

3. Let A be a square matrix with $\det A = 0$. Show that 0 must be an eigenvalue of A.

4. Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 & 5 \\ 0 & 2 & 6 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of A, and give the algebraic multiplicity of each eigenvalue.

(b) Find the Jordan form J of A, and a transition matrix Q, such that $J = Q^{-1}AQ$.

5. Without using the Jordan form algorithm, decide whether each of the following matrices is diagonalizable. Give a **brief** explanation for your answer.

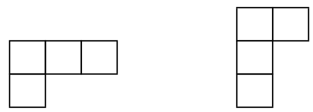
(a) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

6. A certain 8×8 matrix B has two distinct eigenvalues, $\lambda = 2$ and $\lambda = 4$. After performing the Jordan form algorithm, you find the following diagrams:

$\lambda = 2$ $\quad\quad$ $\lambda = 4$



Write down a possible Jordan form for the matrix B.

7. Apply the Gram-Schmidt process to the set

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}; \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 9 \\ 1 \\ 6 \end{bmatrix} \right\}$$

to get an orthonormal basis for $\mathbb{R}^3$.

8. Find values for $b, c \in \mathbb{C}$ that make the matrix

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & b \\ \frac{\sqrt{2}}{2} i & c \end{bmatrix}$$

unitary.

9. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2i \\ -1 \end{bmatrix} \in \mathbb{C}^3$.

(a) Let W be the subspace of all vectors orthogonal to $\mathbf{v}$. Find a basis for W and give the dimension.

(b) The following set of vectors is an ordered orthonormal basis for $\mathbb{C}^3$:

$$B = \left\{ \mathbf{u}_1 = \begin{bmatrix} \frac{1+i}{2} \\ 0 \\ \frac{1-i}{2} \end{bmatrix}; \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \mathbf{u}_3 = \begin{bmatrix} -\frac{i}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \right\}.$$

Express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

(c) Find the coordinates $c_B(\mathbf{v})$.