Homework 10 – Solutions

1. Find the general solution to the equation system

$$\begin{pmatrix} i & 0 & 1 & 2 \\ 0 & 1+i & 2 & 4 \\ 0 & 1 & i & 2+i \end{pmatrix} v = 0$$

where $v \in \mathbb{C}^4$.
Solution: Just compute the row–echelon form as always. The only tricky part is that we have to compute $\frac{1}{i} = -i$ and $\frac{1}{1+i} = \frac{(1-i)}{(1+i)(1-i)} = \frac{1-i}{2}$.

2. Find the eigenvalues and the corresponding eigenvectors of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & 2i \end{pmatrix}.$$

Solution: The characteristic polynomial is

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 0-\lambda & 0 & 0 \\ 0 & 0 & 2i-\lambda & 0 \\ 0 & 0 & 0 & 2i-\lambda \end{pmatrix} = (1-\lambda)\lambda(2i-\lambda)(2i-\lambda).$$

So the eigenvalues are $1, 0, 2i$. The eigenvectors corresponding to $\lambda_1 = 1$ are scalar multiples of e_1, the eigenvectors corresponding to $\lambda_1 = 0$ are scalar multiples of e_2, and the eigenvectors corresponding to $\lambda_1 = 1$ are linear combinations of e_3 and e_4,

3. Let A be a 4×4–matrix and v_1, v_2, v_3, v_4 a basis for $\mathbb{R}^4$ such that v_1 is an eigenvector for A corresponding to the eigenvalue -2 and v_2 is an eigenvector for A corresponding to the eigenvalue $i-2$. Let $B = \begin{pmatrix} v_1 & v_2 & v3 & v_4 \end{pmatrix}$. What can you say about $B^{-1}AB$?
Solution: I claim that

$$B^{-1}AB = \begin{pmatrix} -2 & 0 & * & * \\ 0 & i-2 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

where $*$ can be any number. Why is that? Let's look at the first column of $B^{-1}AB$, it equals

$$B^{-1}ABe_1 = B^{-1}Av_1 = B^{-1}2v_1 = 2B^{-1}v_1.$$

But $v_1 = Be_1$, hence $2B^{-1}v_1 = 2e_1$. Similarly we can compute the second column.

4. Let $A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$.

   (a) Find the eigenvalues of A.
   (b) Find the eigenvectors corresponding to the eigenvalues.
   (c) Find a matrix B such that $B^{-1}AB$ is a diagonal matrix.
   (d) Find A^{100}.

Solution:

(a) The characteristic polynomial is

$$\det \begin{pmatrix} 2-\lambda & -3 \\ -3 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 9.$$

Its two zeros are $\lambda_1 = -1, \lambda_2 = 5$. So -1 and 5 are the two eigenvalues of A.

(b) For $\lambda_1 = -1$ we compute

$$\begin{pmatrix} 2-\lambda_1 & -3 \\ -3 & 2-\lambda_1 \end{pmatrix} = \begin{pmatrix} 2+1 & -3 \\ -3 & 2+1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}.$$

It's now easy to see that the null space is given by $\mu \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For any $\mu \neq 0$ these are eigenvectors. Similarly for $\lambda = 5$ one can show that $\mu \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors for $\mu \neq 0$. (Remember: the zero vector is NOT an eigenvector).

(c) We take $B = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ where v_1 and v_2 are any eigenvectors for λ_1 and λ_2. So for example we can take

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $B^{-1}AB = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$. This can either be seen using a direct computation, or much simpler, we know from the lectures that if B is given by a basis of eigenvectors, then $B^{-1}AB$ is a diagonal matrix and the numbers on the diagonal are the eigenvectors corresponding to v_1 and v_2.

(d) It's very easy to compute $(B^{-1}AB)^{100}$ so we do the following trick:

$$A^{100} = BB^{-1}A^{100}BB^{-1} = B(B^{-1}AB)^{100}B^{-1} = B \begin{pmatrix} (-1)^{100} & 0 \\ 0 & 5^{100} \end{pmatrix} B^{-1}.$$

Note that we used the fact that $B^{-1}A^nB = (B^{-1}AB)^n$.

5. Show that if v is an eigenvector of the matrix A to the eigenvalue λ, then v is an eigenvector of the matrix $A^2 = AA$ to the eigenvalue λ^2.
Solution: We have to show that $A^2v = \lambda^2 v$. We compute

$$A^2v = A \cdot Av = A(Av) = A(\lambda v) = \lambda Av = \lambda\lambda v = \lambda^2 v.$$