Homework 11 – Solutions

3. p. 303, problem 10 (remember what numbers and properties matrices A and $P^{-1}AP$ have in common).
   Solution: Note that $\det(A) = \det(P^{-1}AP)$, so if two matrices are similar, they have to have the same determinant. This is clearly not the case here.

4. Let A be a 3×3–matrix. Assume that a calculation shows that in particular $\lambda = 1$ and $\lambda = 2 + i$ are eigenvalues. Show that A is diagonalizable.
   Solution: We showed in class that if λ is an eigenvalue, then its complex conjugate $\bar{\lambda}$ is an eigenvalue as well. In this situation this shows that $\overline{2+i} = 2 - i$ is an eigenvalue for A. But this means that we have three distinct eigenvalues, but we showed that in that case A is diagonalizable.

5. Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a, b, c \in \mathbb{R}$. Show that A always has two REAL eigenvalues.
   Solution: We compute the characteristic polynomial

$$\det(A - \lambda \mathrm{id}) = \det \begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix} = (a-\lambda)(c-\lambda) - b^2 = \lambda^2 + (-a-c)\lambda - b^2.$$

   But now it's easy to see that the discriminant is never negative, i.e. we always have two real zeros.

6. Let $h(k)$ denote the number of hares in the year k, and $r(k)$ the number of rabbits in the year k. Assume that $h(0) = 2$ and $r(0) = 100$. Assume that the number of foxes and rabbits changes over the years according to the following rule:

$$\begin{aligned} r(k+1) &= 10r(k) - 3h(k) \\ h(k+1) &= 4r(k) + 2h(k). \end{aligned}$$

   What is the number of hares and rabbits after 2 years? After 20 years?
   Solution: We can write

$$\begin{pmatrix} r(k+1) \\ h(k+1) \end{pmatrix} = \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} r(k) \\ h(k) \end{pmatrix}.$$

   We write $A = \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix}$. Note that

$$\begin{pmatrix} r(k) \\ h(k) \end{pmatrix} = A \begin{pmatrix} r(k-1) \\ h(k-1) \end{pmatrix} = \cdots = A^k \begin{pmatrix} r(0) \\ h(0) \end{pmatrix} = A^k \begin{pmatrix} 100 \\ 2 \end{pmatrix}.$$

So we have to compute powers of A. The eigenvalues are $\lambda_1 = 10$ and $\lambda_2 = 2$. So A is diagonalizable. So we can find P as usual such that $P^{-1}AP = D$ is diagonal. And we can compute

$$A^k = (PP^{-1}APP^{-1})^k = P(P^{-1}AP)^k P^{-1}.$$

7. Let $A = \begin{pmatrix} 3 & 1 \\ -9 & -3 \end{pmatrix}$. Find a matrix $P = (v_1\, v_2)$ such that $P^{-1}AP$ is of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Note that v_1 is an eigenvector.
Solution: We compute the characteristic polynomial

$$\det(A - \lambda \mathrm{id}) = \begin{pmatrix} 3-\lambda & 1 \\ -9 & -3-\lambda \end{pmatrix} = \lambda^2.$$

So the only eigenvalue is $\lambda = 0$. An eigenvector is given by a solution to $Av = 0$. For example we could take $v = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Now let $v_1 = v$ and v_2 be any other vector linearly independent from v_1, e.g. $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then an easy calculation shows that with $P = \begin{pmatrix} 1 & 1 \\ -3 & 0 \end{pmatrix}$ the matrix $P^{-1}AP$ is of the required form.