

## Homework 12 – Solutions

1. Find a Jordan matrix  $J$  and a transition matrix  $Q$  such that  $J = Q^{-1}AQ$  for the matrix

$$A = \begin{pmatrix} 5 & -9 \\ 4 & -7 \end{pmatrix}.$$

Solution: It's easy to see that the characteristic polynomial equals  $(\lambda + 1)^2$ , so the only eigenvalue is  $\lambda = -1$  with algebraic multiplicity two. Then

$$E_{-1}^1 = \ker\{A - \lambda \text{id}\} = \ker \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

In particular  $\dim(E_{-1}^1) = 1$ , which is less than the algebraic multiplicity. So we have to continue. We compute

$$E_{-1}^2 = \ker\{(A - \lambda \text{id})^2\} = \ker \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \ker \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2.$$

Now  $\dim(E_{-1}^2) = 2$ , i.e. it equals the algebraic multiplicity. The box diagram has one column with two boxes:



In the bottom box we put a vector in  $E_{-1}^2 = \mathbb{R}^2$  which does not lie in  $E_{-1}^1$ . For example we could take  $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (but there are many other possibilities!). In the box above it we put  $(A - \lambda \text{id})w = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ . So we get

$$\begin{array}{|c|} \hline (A - \lambda \text{id})w \\ \hline w \\ \hline \end{array}$$

Now read off the two vectors from top to bottom, and that's  $J$ , i.e.

$$J = \begin{pmatrix} 6 & 1 \\ 4 & 0 \end{pmatrix}.$$

We get

$$J^{-1}AJ = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

(note that we knew this result even without computing  $J^{-1}AJ$  just from looking at the box diagram for  $\lambda = -1$ ).

2. The following matrix  $A$  has  $\lambda = 1$  as an eigenvalue.

(a) Find the other eigenvalues of  $A$ .

Solution: The characteristic polynomial is  $(1 - \lambda)(\lambda - 1)^2$ , i.e. the only eigenvalue if  $\lambda = 1$  with algebraic multiplicity equal to three. If you couldn't factor the characteristic polynomial immediately, you can take whatever you computed and divide by  $(\lambda - 1)$  (since you know that  $\lambda = 1$  is a zero) and then you get a polynomial of degree two which you can easily factor.

(b) Find  $Q$  and  $J$  as in the previous problem:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 2 & 0 \end{pmatrix}.$$

Solution: We compute

$$A - \lambda \text{id} = A - \text{id} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix}.$$

So  $\dim(E_1^1) = \dim(A - \lambda \text{id}) = 1$ . We furthermore compute

$$(A - \lambda \text{id})^2 = (A - \text{id})^2 = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix}.$$

So  $\dim(E_1^2) = \dim(A - \lambda \text{id}) = 2$ . Since this still doesn't equal the algebraic multiplicity we have to go one further and compute

$$(A - \lambda \text{id})^3 = (A - \text{id})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So now  $E_1^3 = \mathbb{R}^3$ , its dimension equals the algebraic multiplicity. The box diagram has one column with three boxes:



In the bottom box we put a vector in  $E_1^3 = \mathbb{R}^3$  which does not lie in  $E_1^2$ , i.e. a vector in  $\mathbb{R}^3$  which multiplied by  $(A - \text{id})^2$  does not become zero. For example we could take  $w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  (but there are many other possibilities!). In the box above it we put

$(A - \lambda \text{id})w$ , and in the top box we put  $(A - \lambda \text{id})^2 w$ . Then read  $J$  off from the box diagram by going from top to bottom. We get

$$J^{-1}AJ = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that we knew this result even without computing  $J^{-1}AJ$  just from looking at the box diagram for  $\lambda = 1$ , since a column with three boxes stands for a  $3 \times 3$ -Jordan block.

3. Find  $Q$  and  $J$  as above for the matrix  $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ -2 & -3 & -2 \end{pmatrix}$ .

Solution: The characteristic polynomial is  $(2 + \lambda)^2(-\lambda)$ . So we have eigenvalues  $\lambda = -2$  (with algebraic multiplicity two) and  $\lambda = 0$  (with alg. mult. one).

Start with  $\lambda = 0$ . Since it has alg. mult. we know that we only have to find an eigenvector, which in this case is  $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$  (or a multiple of it).

Now turn to  $\lambda = -2$ . We compute

$$A - \lambda \text{id} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ -2 & -3 & 0 \end{pmatrix}.$$

Clearly  $E_{-2}^1 = \ker(A - \lambda \text{id}) = \mathbb{R} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . So its dimension is one, less than the algebraic multiplicity. So we have to compute

$$(A - \lambda \text{id})^2 = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ -2 & -3 & 0 \end{pmatrix}^2 = \begin{pmatrix} 4 & 4 & 0 \\ 0 & 0 & 0 \\ -4 & -4 & 0 \end{pmatrix}.$$

Then the nullspace  $E_{-2}^2$  is two-dimensional. So we stop since that's the algebraic multiplicity. We get a box diagram for  $\lambda = -2$  which has one column with two rows. In the bottom we put a

vector in  $E_{-2}^2$  which does not lie in  $E_{-2}^1$ , we could take  $w = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ . (note that  $(A - \lambda \text{id})w \neq 0$ ,

hence  $w$  does not lie in  $E_{-2}^1$ ). In the box above it put  $(A - \lambda \text{id})w = (A - (-2)\text{id})w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .

Now we take the vectors for  $\lambda = 0$  and  $\lambda = -2$  and get

$$J = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Then

$$J^{-1}AJ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

Note that the first vector in  $J$  is an eigenvector for  $\lambda = 0$ , so the first column of  $J^{-1}AJ$  is completely zero. The last two columns in  $J$  correspond to  $\lambda = -2$ , so the last two columns in  $J^{-1}AJ$  consist of a Jordan block of size two for  $\lambda = -2$ .

4. Let  $A$  be  $2 \times 2$ -matrix of the form  $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ . Show that  $A$  is diagonalizable if and only if  $b = 0$ .

NOTE: I made a mistake here, the problem should have been: Let  $A$  be  $2 \times 2$ -matrix of the form  $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ . Show that  $A$  is diagonalizable if and only if  $b = 0$  or  $a \neq c$ .

Solution: If  $a \neq c$ , then  $A$  has two *different* eigenvalues, namely  $a$  and  $c$ , hence  $A$  is diagonalizable. If  $a = c$ , then  $\dim(\ker(A - aI)) = \dim \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ . This equals two if and only if  $c = 0$ . But only if  $\dim(\ker(A - aI))$  equals the algebraic multiplicity of  $a$  can we conclude that  $A$  is diagonalizable.

5. Let  $A$  be a  $2 \times 2$ -matrix which is NOT diagonalizable. Show that  $A^2$  is not diagonalizable either. (Hint: reduce this problem to an upper triangular matrix using the Jordan normal form).

Solution: It's clear that  $A$  is diagonalizable if and only if  $Q^{-1}AQ$  is diagonalizable for a matrix  $Q$ . But if  $A$  is not diagonalizable, then its Jordan form  $Q^{-1}AQ$  is like in the previous problem, and hence  $Q^{-1}A^2Q = (Q^{-1}AQ)^2$  is also like in the previous problem, so  $Q^{-1}A^2Q$  is not diagonalizable, hence  $A^2$  is not diagonalizable.

8. Suppose  $A$  is a  $7 \times 7$  matrix with three distinct eigenvalues:  $\lambda = 1$  has algebraic multiplicity 3 and geometric multiplicity 2,  $\lambda = 0$  has algebraic multiplicity 2 and geometric multiplicity 1, and  $\lambda = -2$  has algebraic multiplicity 2 and geometric multiplicity 2. Give a Jordan form matrix  $J$  for  $A$ . You **do not** have to find a matrix  $Q$ .

Solution:

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

I'll just do  $\lambda = 1$  here for the justification: The fact that the geometric multiplicity is two means that  $E_{\lambda=1}^1$  has dimension two, so the box diagram has two boxes in the first row. There's one box left, since the algebraic multiplicity is three, so there has to be one box in the second row. But now looking at the columns (which determine the size of the Jordan blocks), we see one column with two boxes, and one column with one box. So there's one Jordan block for  $\lambda = 1$  of size two, and one Jordan block for  $\lambda = 1$  of size one. That's exactly the picture in the first three columns.