2. Let $v_1 = (1, 2, 0, 4)^t$, $v_2 = (0, 1, 0, 1)^t$.

(a) Show that $V := \{v \in \mathbb{R}^4 \text{ orthogonal to } v_1 \text{ and } v_2 \}$ is a subspace of $\mathbb{R}^4$.

Solution: Clearly the zero vector lies in $V$. Now let $v, w \in V$, i.e., $v \cdot v_i = 0$ and $w \cdot v_i = 0$ for $i = 1, 2$. Then we have to show that $v + w \in V$ and $\lambda v \in V$ for any scalar $\lambda$. To show that $v + w \in V$ we have to check that $(v + w) \cdot v_i = 0$ for $i = 1, 2$. But we compute

$$(v + w) \cdot v_i = v \cdot v_i + w \cdot v_i = 0 + 0 = 0.$$  

Similarly $\lambda v \in V$.

(b) Find a basis for $V$.

Solution: Clearly $V$ is the set of all vectors $w = (w_1, w_2, w_3, w_4)$ such that $w \cdot v_1 = 0$ and $w \cdot v_2 = 0$, i.e., such that

$$w_1 + 2w_2 + 0w_3 + 4w_4 = 0, \\
0w_1 + w_2 + 0w_3 + w_4 = 0.$$  

But this means that we just have to solve an equation system! The matrix is

$$
\begin{pmatrix}
1 & 2 & 0 & 4 \\
0 & 1 & 0 & 1
\end{pmatrix}.
$$

Now just do the usual steps: get it into row-echelon form and get a basis vector from each non-leading column.

4. Let $v_1 = (1, 2, 0, 2)^t$, $v_2 = (0, 1, 1, 0)^t$, $v_3 = (0, 0, 1, 0)^t$. Using Gram-Schmidt find an orthogonal basis for the subspace $V$ spanned by $v_1, v_2, v_3$.

Solution:

(a) Compute $u_1 = \frac{1}{||v_1||}v_1 = \begin{pmatrix} 
\frac{1}{\sqrt{3}} \\
\frac{2}{\sqrt{3}} \\
0 \\
0
\end{pmatrix}$. That’s your first new basis vector.

(b) Now compute $u_2 = v_2 - (v_2 \cdot u_1)u_1$, this vector is orthogonal to $u_1$. Then let $u_2 = \frac{1}{||w_2||}w_2$ to get length one.

(c) Now compute $u_3 = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2$, this vector is orthogonal to $u_1$ and $u_2$. Then let $u_3 = \frac{1}{||w_3||}w_3$ to get length one.
6. Let $V$ be the subspace spanned by $v_1 = (1, 2, 2)^t$ and $v_2 = (-2, 2, -1)^t$.

(a) Find an orthonormal basis for $V$.

Solution: Apply Gram Schmidt to get $u_1 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $u_2 = \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$. (note that $v_1$ and $v_2$ are orthogonal, so we just had to normalize them to length one)

(b) Compute the projection of $e_1, e_2, e_3$ onto $V$.

Solution: The projection of $e_i$ onto $V$ is $\pi(e_i) = (e_i \cdot u_1)u_1 + (e_i \cdot u_2)u_2$.

cf. the formula in the notes.

(c) Find the matrix representing the projection $\mathbb{R}^3 \to V$ with respect to the standard basis of $\mathbb{R}^3$ and the basis for $V$ you found in (a).

Solution: We already did all the work! Remember that we get the $i$-th column of the matrix $A$ representing the projection $\pi$ by writing $\pi(i$-basis vector) = $\pi(e_i)$ in terms of the basis $\{u_1, u_2\}$ of $V$. But that’s the computation in (b).

7. We know that if $v_1, \ldots, v_k \in V$ is an orthonormal basis for $V \subset \mathbb{R}^l$, then given any $w \in V$ we can write

$$w = (v_1 \cdot w)v_1 + \ldots + (v_k \cdot w)v_k.$$ 

Put differently, using the inner product we can easily write $w$ as a linear combination of $v_1, \ldots, v_k$. Now assume that $v_1, \ldots, v_k \in V$ is an orthogonal basis for $V \subset \mathbb{R}^l$ (i.e. we no longer know that $|v_i| = 1$). How can we still write $w$ as a linear combination of $v_1, \ldots, v_k$ using the inner product?

Solution: Let $u_i = \frac{1}{|v_i|} v_i$ for $i = 1, \ldots, k$. Then clearly $u_1, \ldots, u_k$ is an orthonormal basis for $V$. So we have

$$w = (u_1 \cdot w)u_1 + \ldots + (u_k \cdot w)u_k.$$ 

Now rewriting this in terms of $v_i$ we get

$$w = \frac{1}{|v_1|^2} (v_1 \cdot w)v_1 + \cdots + \frac{1}{|v_k|^2} (v_k \cdot w)v_k.$$