Homework 13 – Solutions

2. Let $v_1 = (1,2,0,4)^t, v_2 = (0,1,0,1)^t$.
   (a) Show that $V := \{v \in \mathbb{R}^4$ orthogonal to v_1 and $v_2\}$ is a subspace of $\mathbb{R}^4$.
   Solution: Clearly the zero vector lies in V. Now let $v, w \in V$, i.e. $v \cdot v_i = 0$ and $w \cdot v_i = 0$ for $i = 1, 2$. Then we have to show that $v + w \in V$ and $\lambda v \in V$ for any scalar λ. To show that $v + w \in V$ we have to check that $(v + w) \cdot v_i = 0$ for $i = 1, 2$. But we compute

   $$(v + w) \cdot v_i = v \cdot v_i + w \cdot v_i = 0 + 0 = 0.$$

   Similarly $\lambda v \in V$.

   (b) Find a basis for V.
   Solution: Clearly V is the set of all vectors $w = (w_1, w_2, w_3, w_4)$ such that $w \cdot v_1 = 0$ and $w \cdot v_2 = 0$, i.e. such that

   $$\begin{aligned} w_1 + 2w_2 + 0w_3 + 4w_4 &= 0, \\ 0w_1 + w_2 + 0w_3 + w_4 &= 0. \end{aligned}$$

   But this means that we just have to solve an equation system! The matrix is

   $$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

   Now just do the usual steps: get it into row–echelon form and get a basis vector from each non–leading column.

4. Let $v_1 = (1,2,0,2)^t, v_2 = (0,1,1,0)^t, v_3 = (0,0,1,0)^t$. Using Gram-Schmidt find an orthogonal basis for the subspace V spanned by v_1, v_2, v_3.
   Solution:

   (a) Compute $u_1 = \frac{1}{|v_1|} v_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix}$. That's your first new basis vector.

   (b) Now compute $w_2 = v_2 - (v_2 \cdot u_1)u_1$, this vector is orthogonal to u_1. Then let $u_2 = \frac{1}{|w_2|} w_2$ to get length one.

   (c) Now compute $w_3 = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2$, this vector is orthogonal to u_1 and u_2. Then let $u_3 = \frac{1}{|w_3|} w_3$ to get length one.

6. Let V be the subspace spanned by $v_1 = (1, 2, 2)^t$ and $v_2 = (-2, 2, -1)^t$.

   (a) Find an orthonormal basis for V.

   Solution: Apply Gram Schmidt to get $u_1 = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}, u_2 = \frac{1}{3}\begin{pmatrix}-2\\2\\-1\end{pmatrix}$. (note that v_1 and v_2 are orthogonal, so we just had to normalize them to length one)

   (b) Compute the projection of e_1, e_2, e_3 onto V.
   Solution: The projection of e_i onto V is

$$\pi(e_i) = (e_i \cdot u_1)u_1 + (e_i \cdot u_2)u_2,$$

   cf. the formula in the notes.

   (c) Find the matrix representing the projection $\mathbb{R}^3 \to V$ with respect to the standard basis of $\mathbb{R}^3$ and the basis for V you found in (a).
   Solution: We already did all the work! Remember that we get the i–th column of the matrix A representing the projection π by writing $\pi(i\text{–basis vector}) = \pi(e_i)$ in terms of the basis $\{u_1, u_2\}$ of V. But that's the computation in (b).

7. We know that if $v_1, \dots, v_k \in V$ is an *orthonormal* basis for $V \subset \mathbb{R}^l$, then given any $w \in V$ we can write

$$w = (v_1 \cdot w)v_1 + \dots (v_k \cdot w)v_k.$$

   Put differently, using the inner product we can easily write w as a linear combination of $v_1, \dots, v_k$. Now assume that $v_1, \dots, v_k \in V$ is an *orthogonal* basis for $V \subset \mathbb{R}^l$ (i.e. we no longer know that $|v_i| = 1$). How can we still write w as a linear combination of $v_1, \dots, v_k$ using the inner product?
   Solution: Let $u_i = \frac{1}{|v_i|}v_i$ for $i = 1, dots, k$. Then clearly $u_1, \dots, u_k$ is an *orthonormal* basis for V. So we have

$$w = (u_1 \cdot w)u_1 + \dots (u_k \cdot w)u_k.$$

   Now rewriting this in terms of v_i we get

$$w = \frac{1}{|v_1|^2}(v_1 \cdot w)v_1 + \dots + \frac{1}{|v_k|^2}(v_k \cdot w)v_k.$$