Homework 6 – Solutions

1. Consider the following vectors:

$$v_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, v_2 = \begin{pmatrix}3\\2\\5\\0\end{pmatrix}, v_3 = \begin{pmatrix}0\\1\\0\\2\end{pmatrix}, v_4 = \begin{pmatrix}5\\1\\7\\-2\end{pmatrix}, v_5 = \begin{pmatrix}3\\3\\5\\2\end{pmatrix}$$

   (a) Check whether $(6\ 3\ 9\ -2)^t$ lies in the span of $v_1, \ldots, v_5$.
   Solution: We have to check whether $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 + \lambda_5 v_5 = (6\ 3\ 9\ -2)^t$ has a solution. So we have to consider the augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 3 & 0 & 5 & 3 & 6 \\ 0 & 2 & 1 & 1 & 3 & 3 \\ 1 & 5 & 0 & 7 & 5 & 9 \\ 0 & 0 & 2 & -2 & 2 & -2 \end{array}\right),$$

   and as always get it into row–echelon form and check whether there's a solution.

   (b) Show that $v_1, \ldots, v_5$ are linearly dependent.
   Solution: We have to show that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 + \lambda_5 v_5 = 0$ has infinitely many solutions. So we have to consider the augmented matrix

$$\begin{pmatrix} 1 & 3 & 0 & 5 & 3 \\ 0 & 2 & 1 & 1 & 3 \\ 1 & 5 & 0 & 7 & 5 \\ 0 & 0 & 2 & -2 & 2 \end{pmatrix},$$

   and get it into row–echelon form (something you already did in (a)!). You will see that there's at least one non–leading column, hence $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 + \lambda_5 v_5 = 0$ has infinitely many solutions.

   (c) Find a basis for $\mathbb{R}^4$ which contains as many vectors from $\{v_1, \ldots, v_5\}$ as possible.
   Solution: We know that v_1, v_2, v_3, v_4, v_5 are NOT a basis. In fact they don't even span $\mathbb{R}^4$ since the row echelon form has a zero row. So we throw in the vectors e_1, e_2, e_3, e_4, then $v_1, v_2, v_3, v_4, v_5, e_1, e_2, e_3, e_4$ certainly span $\mathbb{R}^4$. But now we have way too many vectors, so we have to throw out some of them. We consider the big matrix

$$\begin{pmatrix} 1 & 3 & 0 & 5 & 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 5 & 0 & 7 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -2 & 2 & 0 & 0 & 0 & 1 \end{pmatrix},$$

get this matrix into row echelon form and take the vectors among $v_1, v_2, v_3, v_4, v_5, e_1, e_2, e_3, e_4$ which correspond to the four leading columns.