2. Let \( B := \{v_1; v_2; v_3\} \) be an ordered basis for a vector space \( V \). What is \( c_B(0), c_B(-v_2), c_B(v_1 + v_3) \)?

Answer: \( c_B(0) = (0, 0, 0), c_B(-v_2) = (0, -1, 0), c_B(v_1 + v_3) = (1, 0, 1) \).

4. Let \( B = \{v_1; v_2; v_3\} \) be an ordered basis. Let \( \tilde{B} := \{v_1 + v_2; v_3; v_2\} \). What is the base change matrix from \( B \) to \( \tilde{B} \), i.e. what is the matrix \( M \) such that

\[ c_B(v) = M c_{\tilde{B}}(v) \]

for all \( v \)?

Recall that \( M \) is given by the coefficients of expressing \( \tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \) in terms of \( v_1, v_2, v_3 \) and then taking the transpose:

\[
\begin{align*}
\tilde{v}_1 &= 1 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 \\
\tilde{v}_2 &= 0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 \\
\tilde{v}_3 &= 0 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3.
\end{align*}
\]

So

\[ M = \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}^t = \begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}. \]

5. (a) Let \( B := \{v_1; v_2; v_3\} \) be an ordered basis for a vector space \( V \). Use the principle of isomorphism to show that \( v_1 + v_2 - v_3 \) and \( v_1 + 2v_2 + v_3 \) are linearly independent.

The idea is to find \( c_B(v_1 + v_2 - v_3) \) and \( c_B(v_1 + 2v_2 + v_3) \) first. Then by the principle of isomorphism we only have to check whether the resulting vectors in \( \mathbb{R}^3 \) are linearly independent:

\[
c_B(v_1 + v_2 - v_3) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, c_B(v_1 + 2v_2 + v_3) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.
\]

Combining these two column vectors we get a \( 3 \times 2 \)-matrix. Now compute the row–echelon form and we see that both columns are leading columns. Therefore the vectors in \( \mathbb{R}^3 \) are linearly independent. By the principle of isomorphism we get that \( v_1 + v_2 - v_3 \in V \) and \( v_1 + 2v_2 + v_3 \in V \) are linearly independent as well.
6. Are the polynomials \( 1 + t + t^3, 1 - 2t + 3t^3, 2 - t + t^3 \) linearly independent in the vector space \( V := \{ \text{polynomials of degree less or equal than 3} \} \).

The idea is to pick an ordered basis for \( V \), then compute the coordinate vectors and check linear independence for the coordinate vectors.

The easiest ordered basis for \( V \) is \( B = \{ 1; t; t^2; t^3 \} \). Then

\[
c_B(1 + t + t^3) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad c_B(1 - 2t + 3t^3) = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \quad c_B(2 - t + t^3) = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.
\]

Now check linear independence of these vectors using the row–echelon form method.