Homework 7 – Solutions

2. Let $B := \{v_1; v_2; v_3\}$ be an ordered basis for a vector space V. What is $c_B(0), c_B(-v_2), c_B(v_1+v_3)$?
Answer: $c_B(0) = (0,0,0), c_B(-v_2) = (0,-1,0), c_B(v_1+v_3) = (1,0,1)$.

4. Let $B = \{v_1; v_2; v_3\}$ be an ordered basis. Let $\tilde{B} := \{v_1+v_2; v_3; v_2\}$. What is the base change matrix from B to $\tilde{B}$, i.e. what is the matrix M such that

$$c_B(v) = M c_{\tilde{B}}(v)$$

for all v?

Recall that M is given by the coefficients of expressing $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ in terms of v_1, v_2, v_3 and then taking the transpose:

$$\begin{array}{rcllllll} \tilde{v}_1 &=& 1\cdot & v_1 & + & 1\cdot & v_2 & + \; 0\cdot \; v_3 \\ \tilde{v}_2 &=& 0\cdot & v_1 & + & 0\cdot & v_2 & + \; 1\cdot \; v_3 \\ \tilde{v}_3 &=& 0\cdot & v_1 & + & 1\cdot & v_2 & + \; 0\cdot \; v_3. \end{array}$$

So

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

5. (a) Let $B := \{v_1; v_2; v_3\}$ be an ordered basis for a vector space V. Use the principle of isomorphism to show that $v_1 + v_2 - v_3$ and $v_1 + 2v_2 + v_3$ are linearly independent.

The idea is to find $c_B(v_1 + v_2 - v_3)$ and $c_B(v_1 + 2v_2 + v_3)$ first. Then by the principle of isomorphism we only have to check whether the resulting vectors in $\mathbb{R}^3$ are linearly independent:

$$c_B(v_1 + v_2 - v_3) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, c_B(v_1 + 2v_2 + v_3) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Combining these two column vectors we get a 3×2–matrix. Now compute the row–echelon form and we see that both columns are leading columns. Therefore the vectors in $\mathbb{R}^3$ are linearly independent. By the principle of isomorphism we get that $v_1 + v_2 - v_3 \in V$ and $v_1 + 2v_2 + v_3 \in V$ are linearly independent as well.

6. Are the polynomials $1+t+t^3, 1-2t+3t^3, 2-t+t^3$ linearly independent in the vector space

$$V := \{\text{polynomials of degree less or equal than } 3\}.$$

The idea is to pick an ordered basis for V, then compute the coordinate vectors and check linear independence for the coordinate vectors.
The easiest ordered basis for V is $B = \{1; t; t^2; t^3\}$. Then

$$c_B(1+t+t^3) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, c_B(1-2t+3t^3) = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}, c_B(2-t+t^3) = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now check linear independence of these vectors using the row–echelon form method.