Homework 8 – solutions

3. Let $V = W$ be the vector space of all differentiable functions defined on $[0, 1]$. Which of the following maps $\varphi : V \to W$ is a linear transformation?

   (a) $\varphi(p(x)) = p(x)$ where $p(x) \in V$. YES.

   (b) $\varphi(p(x)) = \sin(x) \cdot p(x)$ where $p(x) \in V$. YES.

   (c) $\varphi(p(x)) = p(x)^2$ where $p(x) \in V$. NO, since $\varphi(2p(x)) = 4p(x)^2 \neq 2\varphi(p(x))$.

   (d) $\varphi(p(x)) = 3p(x) + 4$ where $p(x) \in V$. NO, since e.g. $\varphi(1 + 1) = \varphi(2) = 10$ but $\varphi(1) + \varphi(1) = 7 + 7 = 14$.

   (e) $\varphi(p(x)) = 0$ where $p(x) \in V$. YES.

   (f) $\varphi(p(x)) = \int_0^x p(t)\,dt$ where $p(x) \in V$ (note that the right hand side is again a function!). YES.

4. Let $V = W = \mathbb{R}^2$. Determine which of the following are linear transformations.

   (a) Translation by a non–zero vector. NO, for example the zero vector always has to become the zero vector under a linear transformation, that's not the case here.

   (b) Rotation by an angle α. YES.

   (c) Reflection about the x–axis. YES.

   (d) Stretching by a factor of two. YES.

5. Let V be $\mathbb{R}^3$ and $W = \mathbb{R}^2$. Then consider the map $\varphi : V \to W$ defined by $\varphi(a, b, c) := (0, a + b + c))$.

   (a) Show that φ is a linear transformation.

   (b) Find the kernel of φ.
   Solution: The kernel is given by all $(a, b, c) \in \mathbb{R}^3$ with $a + b + c = 0$. This is an example of a linear equation system, so let's just use our usual approach: The matrix becomes

   $$(\,1\,1\,1\,|0\,)$$

   which is already in row–echelon form. So the solution space is given by

   $$\lambda_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

   This is then the kernel of φ.

(c) Find the image space of φ.
Solution: The image space is

$$\left\{ \begin{pmatrix} 0 \\ a \end{pmatrix} | a \in \mathbb{R} \right\},$$

clearly any $\varphi(v)$ is of that form, and $\begin{pmatrix} 0 \\ a \end{pmatrix} = \varphi\left(\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}\right)$. Put differently, the image space is all vectors in $\mathbb{R}^2$ where the first coordinate is zero.

(d) Verify the formula $\dim(\ker(\varphi)) + \dim(\mathrm{Im}(\varphi)) = \dim(V)$.
Solution: Clearly $\dim(\ker(\varphi)) = 2$, $\dim(\mathrm{Im}(\varphi)) = 1$, $\dim(V) = 3$.

(e) Find a basis for $\ker(\varphi)$.
Solution: We did this in (b), a possible basis is

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(f) Find vectors in V such that their image under φ is a basis for the image space.
Solution: A basis for $\mathrm{Im}(\varphi)$ is given by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So if we take $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then $\varphi(v)$ is a basis for the image space of φ.

(g) Show that the vectors from (5) and (6) put together give a basis for V.
Solution: Take these three vectors, put them into a matrix, find the row–echelon form, you will see that it is the identity matrix.