

Homework 8, due Friday 10/28

1. Write down the definition of a vector space (handwritten).
2. Write down the definition of a linear transformation (handwritten).
3. Let $V = W$ be the vector space of all differentiable functions defined on $[0, 1]$. Which of the following maps $\varphi : V \rightarrow W$ is a linear transformation?
 - (a) $\varphi(p(x)) = p(x)$ where $p(x) \in V$.
 - (b) $\varphi(p(x)) = \sin(x) \cdot p(x)$ where $p(x) \in V$.
 - (c) $\varphi(p(x)) = p(x)^2$ where $p(x) \in V$.
 - (d) $\varphi(p(x)) = 3p(x) + 4$ where $p(x) \in V$.
 - (e) $\varphi(p(x)) = 0$ where $p(x) \in V$.
 - (f) $\varphi(p(x)) = \int_0^x p(t) dt$ where $p(x) \in V$ (note that the right hand side is again a function!).
4. Let $V = W = \mathbb{R}^2$. Determine which of the following are linear transformations.
 - (a) Translation by a non-zero vector.
 - (b) Rotation by an angle α .
 - (c) Reflection about the x -axis.
 - (d) Stretching by a factor of two.
5. Let V be \mathbb{R}^3 and $W = \mathbb{R}^2$. Then consider the map $\varphi : V \rightarrow W$ defined by $\varphi(a, b, c) := (0, a + b + c)$.
 - (a) Show that φ is a linear transformation.
 - (b) Find the kernel of φ .
 - (c) Find the image space of φ .
 - (d) Verify the formula $\dim(\ker(\varphi)) + \dim(\text{Im}(\varphi)) = \dim(V)$.
 - (e) Find a basis for $\ker(\varphi)$.
 - (f) Find vectors in V such that their image under φ is a basis for the image space.
 - (g) Show that the vectors from (5) and (6) put together give a basis for V .
6. Let V be the vector space of all polynomials of degree less or equal than four. Let $W = \mathbb{R}$. Then consider the map $\varphi : V \rightarrow W$ defined by $\varphi(p(t)) = p(2)$. Note that this associates to a polynomial $p(t) \in W$ a number, namely $p(2)$, which is an element of W .

- (a) Show that φ is a linear transformation.
 - (b) Find the kernel of φ .
 - (c) Find the image space of φ .
 - (d) Verify the formula $\dim(\ker(\varphi)) + \dim(\text{Im}(\varphi)) = \dim(V)$.
7. (For experts, it does not count towards your homework grade). Give an example of a map $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi(v + w) = \varphi(v) + \varphi(w)$ for all v, w but not necessarily $\varphi(\lambda v) = \lambda\varphi(v)$.