## Homework 9 – Solutions

1. Let $V$ be the vector space of all differentiable functions on $\mathbb{R}$. Consider $\varphi : V \to V$ given by $\varphi(f(x)) = e^x \cdot f(x)$. This means that to the function $f(x)$ we associate the function $e^x f(x)$.

   (a) Show that $\varphi$ is a linear transformation.
   Solution: Let $f(x), g(x) \in V$, then
   $$\varphi(f(x) + g(x)) = e^x(f(x) + g(x)) = e^x f(x) + e^x g(x) = \varphi(f(x)) + \varphi(g(x))$$
   similarly for $f(x) \in V, \lambda \in \mathbb{R}$ we have
   $$\varphi(\lambda f(x)) = e^x(\lambda f(x)) = \lambda e^x f(x) = \lambda \varphi(f(x)).$$

   (b) Determine the kernel of $\varphi$.
   Solution: We have to find all $f(x)$ such that $e^x f(x) = 0 \in V$, i.e. we have to find all $f(x)$ such that $e^x f(x) = 0$ for all $x$. But since $e^x \neq 0$ for all $x$, it follows that if $e^x f(x) = 0$ for all $x$ then $f(x) = 0$ for all $x$. So $\text{Ker}(\varphi) = 0$.

   (c) Determine whether $\varphi$ is invertible. If $\varphi$ is invertible, what is the inverse.
   Solution: Let $\psi : V \to V$ given by $\psi(f(x)) := \frac{1}{e^x} f(x)$, then
   $$\varphi(\psi(f(x))) = \varphi(\frac{1}{e^x} f(x)) = e^x \frac{1}{e^x} f(x) = f(x),$$
   similarly $\psi(\varphi(f(x)) = f(x)$. Therefore $\psi$ is the inverse of $\phi$.

   (d) Student A says: "let's use matrices to solve (c), just as we did in class". What do you say to this idea?
   Matrices only work if we have a basis, but there is no (finite) basis for $V$. So we can not use this approach.

2. Let $V$ be a vector space with ordered basis $B := \{v_1; v_2\}$ and $W$ a vector space with ordered basis $\tilde{B} := \{w_1, w_2, w_3\}$. Consider the linear transformation $\varphi$ with $\varphi(v_1) = w_1 - w_2 + w_3$ and $\varphi(v_2) = -w_2 + 2w_3$

   (a) What is $c_B(v_1), c_B(v_2)$?
   Solution $c_B(v_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, c_B(v_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   (b) What is $c_{\tilde{B}}(w_1), c_{\tilde{B}}(w_2), c_{\tilde{B}}(w_3)$?
   Solution $c_{\tilde{B}}(w_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, c_{\tilde{B}}(w_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, c_{\tilde{B}}(w_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(c) What is $c_{\tilde{B}}(\varphi(v_1)), c_{\tilde{B}}(\varphi(v_2))$?
Solution:

$$\begin{aligned} c_{\tilde{B}}(\varphi(v_1)) &= c_{\tilde{B}}(w_1 - w_2 + w_3) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ c_{\tilde{B}}(\varphi(v_2)) &= c_{\tilde{B}}(-w_2 + 2w_3) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}. \end{aligned}$$

(d) What is the matrix representing $\varphi$ with respect to the ordered bases $B$ and $\tilde{B}$.
Solution: We get the matrix by expressing $\varphi(v_i)$ in terms of $\tilde{B}$, and then reading of the coefficients and taking the transpose:

$$\begin{aligned} \varphi(v_1) &= w_1 - w_2 + w_3 \\ \varphi(v_2) &= 0w_1 + -w_2 + 2w_3 \end{aligned}$$

so we read of the coefficients

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

and taking the transpose we get the desired matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 2 \end{pmatrix}.$$

Note that this is the same as putting $c_{\tilde{B}}(\varphi(v_1))$ and $c_{\tilde{B}}(\varphi(v_2))$ into a matrix.

4. Let $A$ be a $p \times q$–matrix. Consider the linear transformation $\varphi : \mathbb{R}^q \to \mathbb{R}^p$ given by $\varphi(v) = Av$. Take $\{e_1; \dots; e_p\}$ and $\{e_1; \dots; e_q\}$ as the ordered bases. What is the matrix representing $\varphi$ with respect to these bases.
Solution: Write $A = (a_{ij})$, then taking the $i$-th vector of the basis $\{e_1; \dots; e_q\}$ and writing its image under $\varphi$ in terms of the basis $\{e_1; \dots; e_p\}$ we get

$$\varphi(e_i) = Ae_i = a_{1i}e_1 + a_{2i}e_2 + \dots + a_{pi}e_p$$

so we see that the matrix representing $\varphi$ with the standard basis for $\mathbb{R}^q$ and $\mathbb{R}^p$ is just $A$ itself.

3. Let $V$ be the vector space of all polynomials $p(t)$ of degree less or equal than 3 such that $p(2) = 0$. We take the ordered basis $B := \{t - 2; (t - 2)^2; (t - 2)^3\}$. Let $W$ be the vector space of all polynomials $p(t)$ of degree less or equal than 2. For $W$ we take the ordered basis $C := \{1; t; t^2\}$. Now we take the linear transformation $\varphi : V \to W$ which is defined by $\varphi(p(t)) := p'(t)$.

(a) What is the matrix representing $\varphi$ with respect to the ordered bases $B$ and $C$?
Solution: We compute

$$\begin{aligned} c_C(\varphi(t-2)) &= c_C(1) = \begin{pmatrix}1\\0\\0\end{pmatrix} \\ c_C(\varphi((t-2)^2)) &= c_C(2t-4) = \begin{pmatrix}-4\\2\\0\end{pmatrix} \\ c_C(\varphi((t-2)^3)) &= c_C(3t^2-12t+12) = \begin{pmatrix}12\\-12\\3\end{pmatrix}. \end{aligned}$$

The matrix representing $\varphi$ is just given by the above vectors

$$\begin{pmatrix}1 & -4 & 12\\ 0 & 2 & -12\\ 0 & 0 & 3\end{pmatrix}$$

(b) Does $\varphi$ have an inverse?
Solution: $\varphi$ is invertible if and only if the matrix representing it (with respect to any basis) is invertible. In our case the determinant equals 6, so the matrix and hence $\varphi$ are invertible.

(c) If $\varphi$ has an inverse, what is it? More precisely, what is the map $\psi : W \to V$ such that $(\psi \circ \varphi)(p(t)) = p(t)$ for any $p(t) \in V$?
Solution: Consider the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ c_B \downarrow & & \downarrow c_C \\ \mathbb{R}^3 & \xrightarrow{A} & \mathbb{R}^3 \end{array}$$

So if we want to go from $W$ back to $V$ we can also make a detour and go as follows: Let $w = a + bt + ct^2 \in W$, then

$$c_C(w) = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

Furthermore

$$A^{-1}(c_C(w)) = \begin{pmatrix}1 & 2 & 4\\ 0 & \frac{1}{2} & 2\\ 0 & 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}a+2b+4c\\ \frac{1}{2}b+2c\\ \frac{1}{3}c\end{pmatrix}.$$

Now we have to go from $\mathbb{R}^3$ back to $V$, i.e. we have to compute the inverse map $(c_B)^{-1} : \mathbb{R}^3 \to V$. Given $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in \mathbb{R}^3$ this is given by

$$(c_B)^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \lambda_1 + \lambda_2 t + \lambda_3 t^2.$$

So putting everything together we get

$$\varphi^{-1} = (c_B)^{-1}(A^{-1}(c_C(w))) = (c_B)^{-1} \begin{pmatrix} a + 2b + 4c \\ \frac{1}{2}b + 2c \\ \frac{1}{3}c \end{pmatrix} = (a+2b+4c)(t-2)+(\frac{1}{2}b+2c)(t-2)^2+\frac{1}{3}c(t-2)^3.$$