

Homework 5, due Friday 2/17

1. Let $(X, \{e_\alpha\}_{\alpha \in A})$ and $(Y, \{e_\beta\}_{\beta \in B})$ be CW complexes with finitely many cells in each dimension. Give the space $X \times Y$ the structure of a CW complex, i.e. find a decomposition into open cells such that the axioms are satisfied.
2. Let K be the Klein bottle. More precisely, let K be the result of identifying the corresponding sides of the square given by the vertices $(0, 0), (1, 0), (0, 1), (1, 1)$ (identify the horizontal ones as usual, and the vertical ones by changing the orientation).
 - (a) Write down a CW complex structure for K with one 0-cell, two 1-cells and one 2-cells.
 - (b) Write down characteristic maps for each cell.
 - (c) Compute the homology of the Klein bottle.
3. Give \mathbb{R} the structure of a CW-complex and use it to compute its homology.
4. Give $B^n = \{v \in \mathbb{R}^n \mid \|v\| \leq 1\}$ a CW-complex structure (what's the simplest one?) and use it to compute its homology.
5. Compute the homology of the oriented closed surface of genus g . Recall that the surface is given by the following identifications on the boundary:

[here should be a picture of a $4g$ -gon, where the sides are $a_1 b_1 a'_1 b'_1 a_2 b_2 a'_2 b'_2 \dots$ and the sides a_i and a'_i are identified with opposite directions and the sides b_i and b'_i are identified with opposite directions. In the case of a torus, $g = 1$, you should get the usual picture of a torus as the result of identifying sides of a square]

Note: First make this picture more precise, e.g. by looking at D^2 , its boundary S^1 and writing down how pieces of S^1 get identified. Then write down a CW-structure for the quotient space.

6.
 - (a) Show that for every cyclic group \mathbb{Z}/n there exists a CW complex X with $H_1(X) = \mathbb{Z}/n$ (hint: think carefully about $\mathbb{R}P^2$).
 - (b) Show that every finitely generated abelian group is the first homology of a space.