NON–COMMUTATIVE MULTIVARIABLE REIDEMEISTER TORSION AND THE THURSTON NORM

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ABSTRACT. Given a 3–manifold the second author defined functions $\delta_n : H^1(M;\mathbb{Z}) \to \mathbb{N}$, generalizing McMullen's Alexander norm, which give lower bounds on the Thurston norm. We reformulate these invariants in terms of Reidemeister torsion over a non–commutative multivariable Laurent polynomial ring. This allows us to show that these functions are semi-norms.

1. INTRODUCTION

Let M be a 3-manifold. Throughout the paper we will assume that all 3-manifolds are compact, connected and orientable. Let $\phi \in H^1(M;\mathbb{Z})$. The *Thurston norm* of ϕ is defined as

$$||\phi||_T = \min\{\chi_-(S) \,|\, S \subset M \text{ properly embedded surface dual to } \phi\}$$

where given a surface S with connected components $S_1, \dots, S_k$ we write $\chi_-(S) = \sum_{i=1}^{k} \max\{0, -\chi(S_i)\}$. We refer to [Th86] for details.

Generalizing work of Cochran [Co04] the second author introduced in [Ha05] a function

$$\delta_n : H^1(M;\mathbb{Z}) \to \mathbb{N}_0 \cup \{-\infty\}$$

for every $n \in \mathbb{N}$ and showed that δ_n gives a lower bound on the Thurston norm for every n. These functions are invariants of the 3-manifold and generalize the Alexander norm defined by C. McMullen in [Mc02]. We point out that the definition we use here differs slightly from the original definition when $n = 0$ and a few other special cases. We refer to Section 4.3 for details.

The relationship between the functions δ_n and the Thurston norm was further strengthened in [Ha06] (cf. also [Co04] and [Fr05]) where it was shown that the δ_n give a never decreasing series of lower bounds on the Thurston norm, i.e. for any $\phi \in H^1(M;\mathbb{Z})$ we have

$$\delta_0(\phi) \le \delta_1(\phi) \le \delta_2(\phi) \le \dots \le ||\phi||_T.$$

Furthermore it was shown in [FK05c] that under a mild assumption these inequalities are an equality modulo 2.

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Thurston [Th86] showed in particular that $||-||_T$ is a seminorm. It is therefore a natural question to ask whether the invariants δ_n are seminorms as well. In [Ha05] this was shown to be the case for $n = 0$. The following theorem, which is a special case of the main theorem of this paper (cf. Theorem 4.2), gives an affirmative answer for all n.

Theorem 1.1. *Let M be a 3–manifold with empty or toroidal boundary. Assume that $\delta_n(\phi) \neq -\infty$ for some $\phi \in H^1(M;\mathbb{Z})$, then*

$$\delta_n : H^1(M;\mathbb{Z}) \to \mathbb{N}_0$$

is a seminorm.

This in particular allows us to show that the sequence $\{\delta_n\}$ is eventually constant. That is, there exists an $N \in \mathbb{N}$ such that $\delta_n = \delta_N$ for all $n \geq N$ (cf. Proposition 4.4).

Initially we discuss a more algebraic problem. Recall that given a multivariable Laurent polynomial ring $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ over a commutative field $\mathbb{F}$ we can associate to any non–zero $f = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha \in \mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ a seminorm on $\mathrm{Hom}(\mathbb{Z}^m, \mathbb{R})$ by

$$||\phi||_f := \sup\{\phi(\alpha) - \phi(\beta) | a_\alpha \neq 0, a_\beta \neq 0\}.$$

Furthermore, to any square matrix B over $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ with $\det(B) \neq 0$ we can associate a norm using $\det(B) \in \mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

Generalizing this idea to the non-commutative case, in Section 2.1 we introduce the notion of a *multivariable skew Laurent polynomial ring* $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ of rank m over a skew field $\mathbb{K}$. Given a square matrix B over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ we can study its Dieudonné determinant $\det(B)$ which is an element in the abelianization of the multiplicative group $\mathbb{K}(t_1, \dots, t_m) \setminus \{0\}$ where $\mathbb{K}(t_1, \dots, t_m)$ denotes the quotient field of $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. This determinant will in general not be represented by an element in $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Our main technical result (Theorem 2.2) is that nonetheless there is a natural way to associate a norm to B which generalizes the commutative case.

Given a 3–manifold M and a 'compatible'–representation

$$\pi_1(M) \to \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$$

we will show in Section 3 that the corresponding Reidemeister torsion can be viewed as a matrix over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. We will show in Section 4.3 that for appropriate representations the norm which we can associate to the matrix over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ agrees with δ_n. In particular, this implies Theorem 1.1. We conclude this paper with examples of links for which we compute the Thurston norm using these invariants.

As a final remark we point out that the results in this paper completely generalize the results in [FK05b]. Furthermore the results can easily be extended to studying 2–complexes together with the Turaev norm which is modeled on the definition of the Thurston norm of a 3–manifold. We refer to [Tu02a] for details.

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2. The non–commutative Alexander norm

2.1. **Multivariable Laurent polynomials.** Let $\mathcal{R}$ be a (non–commutative) domain and $\gamma : \mathcal{R} \to \mathcal{R}$ a ring homomorphism. Then we denote by $\mathcal{R}[s^{\pm 1}]$ the *one–variable skew Laurent polynomial ring over* $\mathcal{R}$. Specifically the elements in $\mathcal{R}[s^{\pm 1}]$ are formal sums $\sum_{i=m}^{n} a_i s^i$ $(m \leq n \in \mathbb{Z})$ with $a_i \in \mathcal{R}$. Addition is given by addition of the coefficients, and multiplication is defined using the rule $s^i a = \gamma^i(a) s^i$ for any $a \in \mathcal{R}$ (where $\gamma^i(a)$ stands for $(\gamma \circ \cdots \circ \gamma)(a)$). We point out that any element $\sum_{i=m}^{n} a_i s^i \in \mathcal{R}[s^{\pm 1}]$ can also be written uniquely in the form $\sum_{i=m}^{n} s^i \tilde{a}_i$, indeed, $\tilde{a}_i = s^{-i} a_i s^i \in \mathcal{R}$.

In the following let $\mathbb{K}$ be a skew field. We then define *multivariable skew Laurent polynomial ring of rank* m *over* $\mathbb{K}$ (in non–commuting variables) to be a ring R which is an algebra over $\mathbb{K}$ with unit (i.e. we can view $\mathbb{K}$ as a subring of R) together with a decomposition $R = \oplus_{\alpha \in \mathbb{Z}^m} V_\alpha$ such that the following hold:

(1) V_α is a one–dimensional $\mathbb{K}$–vector space,
(2) $V_\alpha \cdot V_\beta = V_{\alpha+\beta}$,
(3) $V_{(0,\dots,0)} = \mathbb{K}$.

In particular R is $\mathbb{Z}^m$–graded. Note that these properties imply that any V_α is invariant under left and right multiplication by $\mathbb{K}$, that any element in $V_\alpha \setminus \{0\}$ is a unit, and that R is a (non–commutative) domain.

The example to keep in mind is a commutative Laurent polynomial ring $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Let $t^\alpha := t_1^{\alpha_1} \cdot \dots \cdot t_m^{\alpha_m}$ for $\alpha = (\alpha_1, \dots, \alpha_m)$, then $V_\alpha = \mathbb{F} t^\alpha, \alpha \in \mathbb{Z}^m$ has the required properties.

Let R be a multivariable skew Laurent polynomial ring of rank m over $\mathbb{K}$. To make our subsequent definitions and arguments easier to digest we will always pick $t^\alpha \in V_\alpha \setminus \{0\}$ for $\alpha \in \mathbb{Z}^m$. It is easy to see that we can in fact pick $t^\alpha, \alpha \in \mathbb{Z}^m$ such that $t^{n\alpha} = (t^\alpha)^n$ for all $\alpha \in \mathbb{Z}^m$ and $n \in \mathbb{Z}$. Note that this choice in particular implies that $t^{(0,\dots,0)} = 1$. We get the following properties:

(1) $t^\alpha t^{\tilde{\alpha}} t^{-(\alpha+\tilde{\alpha})} \in \mathbb{K}^\times$ for all $\alpha, \tilde{\alpha} \in \mathbb{Z}^m$, and
(2) $t^\alpha \mathbb{K} = \mathbb{K} t^\alpha$ for all α.

This shows that the notion of multivariable skew Laurent polynomial ring of rank m is a generalization of the notion of twisted group ring of $\mathbb{Z}^m$ as defined in [Pa85, p. 13]. If $m = 1$ then we have $t^{(n)} \in V_{(n)}$ such that $t^{(n)} = (t^{(1)})^n$ for any $n \in \mathbb{Z}$. We write $t^n = t^{(n)}$. In particular we have a one–variable skew Laurent polynomial ring as above.

The argument of [DLMSY03, Corollary 6.3] can be used to show that any such Laurent polynomial ring is a (left and right) Ore domain and in particular has a (skew) quotient field. We normally denote a multivariable skew Laurent polynomial ring of rank m over $\mathbb{K}$ suggestively by $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and we denote the quotient field of $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ by $\mathbb{K}(t_1, \dots, t_m)$.

2.2. **The Dieudonné determinant.** In this section we recall several well–known definitions and facts. Let $\mathcal{K}$ be a skew field. In our applications $\mathcal{K}$ will be the quotient field of a multivariable skew Laurent polynomial ring. First define $\mathrm{GL}(\mathcal{K}) := \lim\limits_{\to} \mathrm{GL}(\mathcal{K}, n)$, where we have the following maps in the direct system: $\mathrm{GL}(\mathcal{K}, n) \to \mathrm{GL}(\mathcal{K}, n+1)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, then define $K_1(\mathcal{K}) = \mathrm{GL}(\mathcal{K})/[\mathrm{GL}(\mathcal{K}), \mathrm{GL}(\mathcal{K})]$. For details we refer to [Mi66] or [Tu01].

Let A a square matrix over $\mathcal{K}$. After elementary row operations and destabilization we can arrange that in $K_1(\mathcal{K})$ the matrix A is represented by a 1×1–matrix (d). Then the Dieudonné determinant $\det(A) \in \mathcal{K}^\times_{ab} := \mathcal{K}^\times/[\mathcal{K}^\times, \mathcal{K}^\times]$ (where $\mathcal{K}^\times := \mathcal{K} \setminus \{0\}$) is defined to be d. It is well–known that the Dieudonné determinant induces an isomorphism $\det : K_1(\mathcal{K}) \to \mathcal{K}^\times_{ab}$. We refer to [Ro94, Theorem 2.2.5 and Corollary 2.2.6] for more details.

2.3. **Multivariable skew Laurent polynomial rings and seminorms.** Let $\mathbb{K}[s^{\pm 1}]$ be a one–variable skew Laurent polynomial ring and let $f \in \mathbb{K}[s^{\pm 1}]$. If $f = 0$ then we write $\deg(f) = -\infty$, otherwise, for $f = \sum_{i=m}^{n} a_i s^i \in \mathbb{K}[s^{\pm 1}]$ with $a_m \neq 0, a_n \neq 0$ we define $\deg(f) := n - m$. This extends to a homomorphism $\deg : \mathbb{K}(s) \setminus \{0\} \to \mathbb{Z}$ via $\deg(fg^{-1}) = \deg(f) - \deg(g)$. Since deg is a homomorphism to an abelian group this induces a homomorphism $\deg : \mathbb{K}(s)^\times_{ab} \to \mathbb{Z}$. Note that throughout this paper we will apply the convention that $-\infty < a$ for any $a \in \mathbb{Z}$.

For the remainder of this section let $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ be a multivariable skew Laurent polynomial ring of rank m together with a choice of $t^\alpha, \alpha \in \mathbb{Z}^m$ as above. Let $f \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. We can write $f = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha$ for some $a_\alpha \in \mathbb{K}$. We associate a seminorm $||\_||_f$ on $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R})$ to f as follows. If $f = 0$, then we set $||\_||_f := 0$. Otherwise we set

$$||\phi||_f := \sup\{\phi(\alpha) - \phi(\beta) | a_\alpha \neq 0, a_\beta \neq 0\}.$$

Clearly $||\_||_f$ is a seminorm and does not depend on the choice of t^α. This seminorm should be viewed as a generalization of the degree function.

Now let $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$ and let $f_n, f_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $\det(\tau) = f_n f_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)^\times_{ab}$. Then define

$$||\phi||_\tau := \max\{0, ||\phi||_{f_n} - ||\phi||_{f_d}\}$$

for any $\phi \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R})$. By the following proposition this function is well–defined.

Proposition 2.1. *Let $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$. Let $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $\det(\tau) = f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)^\times_{ab}$. Then*

$$||\_||_{f_n} - ||\_||_{f_d} = ||\_||_{g_n} - ||\_||_{g_d}.$$

We postpone the proof to Section 2.4.

Let B be a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Then it is in general not the case that $\det(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])$ can be represented by an element in $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. But we still have the following result which is the main technical result of this paper.

Theorem 2.2. *If $\tau \in K_1(\mathbb{K}(t_1,\dots,t_m))$ can be represented by a matrix defined over $\mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}]$, then $||\_||_\tau$ defines a seminorm on $Hom(\mathbb{R}^m,\mathbb{R})$.*

We postpone the proof to Section 2.5.

Now let $\phi : \mathbb{Z}^m \to \mathbb{Z}$ be a non–trivial homomorphism. We will show that $||\phi||_B$ can also be viewed as the degree of a polynomial associated to B and ϕ. We begin with some definitions. Consider

$$\mathbb{K}[\mathrm{Ker}(\phi)] := \bigoplus_{\alpha\in Ker(\phi)} \mathbb{K}t^\alpha \subset \mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}].$$

This clearly defines a subring of $\mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}]$ and the argument of [DLMSY03, Corollary 6.3] shows that $\mathbb{K}[\mathrm{Ker}(\phi)]$ is an Ore domain with skew field which we denote by $\mathbb{K}(\mathrm{Ker}(\phi))$.

Let $d \in \mathbb{Z}$ such that $\mathrm{Im}(\phi) = d\mathbb{Z}$ and pick $\beta = (\beta_1,\dots,\beta_m) \in \mathbb{Z}^m$ such that $\phi(\beta) = d$. Let $\mu := t^\beta$. Then we can form one–variable Laurent polynomial rings $(\mathbb{K}[\mathrm{Ker}(\phi)])[s^{\pm1}]$ and $\mathbb{K}(\mathrm{Ker}(\phi))[s^{\pm1}]$ where $sk := \mu k \mu^{-1} s$ for all $k \in \mathbb{K}[\mathrm{Ker}(\phi)]$ respectively for all $k \in \mathbb{K}(\mathrm{Ker}(\phi))$. We get a map

$$\begin{array}{rcl} \gamma_\phi : \mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}] & \xrightarrow{\cong} & (\mathbb{K}[\mathrm{Ker}(\phi)])[s^{\pm1}] \\ \sum_{\alpha\in\mathbb{Z}^m} k_\alpha t^\alpha & \mapsto & \sum_{\alpha\in\mathbb{Z}^m} k_\alpha t^\alpha \mu^{-\phi(\alpha)/d}\, s^{\phi(\alpha)/d}, \end{array}$$

where $k_\alpha \in \mathbb{K}$ for all $\alpha \in \mathbb{Z}^m$. Note that $k_\alpha t^\alpha \mu^{-\phi(\alpha)/d} \in \mathbb{K}[\mathrm{Ker}(\phi)]$. An easy computation shows that γ_ϕ is an isomorphism of rings. Clearly we also get an induced isomorphism $\mathbb{K}(t_1,\dots,t_m) \xrightarrow{\cong} (\mathbb{K}(\mathrm{Ker}(\phi)))(s)$.

Let B a matrix over $\mathbb{K}(t_1,\dots,t_m)$. Define $\deg_\phi(B) := \deg(\det(\gamma_\phi(B)))$ where we view $\gamma(B)$ as a matrix over $\mathbb{K}(\mathrm{Ker}(\phi))(s)$.

Theorem 2.3. *Let B a matrix over $\mathbb{K}(t_1,\dots,t_m)$. Let $\phi \in Hom(\mathbb{Z}^m,\mathbb{Z})$ non–trivial and let $d \in \mathbb{N}$ such that $Im(\phi) = d\mathbb{Z}$. Then*

$$||\phi||_B = d \max\{0, deg_\phi(B)\}.$$

Note that this shows in particular that $\deg_\phi(B)$ is independent of the choice of β. This theorem is a generalization of [Ha05, Proposition 5.12] to the non–commutative case.

Proof. Since γ and deg are homomorphisms it is clearly enough to show that for any $g \in \mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}] \setminus \{0\}$ we have

$$||\phi||_g = d \deg(\gamma_\phi(g)).$$

Write $g = \sum_{\alpha\in\mathbb{Z}^m} a_\alpha t^\alpha$ with $a_\alpha \in \mathbb{K}$. Let d, β, μ and $\gamma : \mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}] \xrightarrow{\cong} (\mathbb{K}[\mathrm{Ker}(\phi)])[s^{\pm1}]$ as above. Note that $\mathrm{Ker}(\phi) \oplus \mathbb{Z}\beta = \mathbb{Z}^m$, hence

$$\begin{array}{rcl} g & = & \sum_{i\in\mathbb{Z}} \sum_{\alpha\in Ker(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta}, \\ \gamma_\phi(g) & = & \sum_{i\in\mathbb{Z}} \Big(\sum_{\alpha\in Ker(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} \Big) s^i. \end{array}$$

Note that $a_{\alpha+i\beta}t^{\alpha+i\beta}\mu^{-i} \subset \mathbb{K}t^{\alpha}$. Since $\mathbb{K}[\mathrm{Ker}(\phi)] = \oplus_{\alpha\in Ker(\phi)}\mathbb{K}t^{\alpha}$ we get the following equivalences:

$$\begin{array}{rrcl} & \sum_{\alpha\in Ker(\phi)} a_{\alpha+i\beta}t^{\alpha+i\beta}\mu^{-i} & = & 0 \\ \Leftrightarrow & a_{\alpha+i\beta}t^{\alpha+i\beta}\mu^{-i} & = & 0 \text{ for all } \alpha \in \mathrm{Ker}(\phi) \\ \Leftrightarrow & a_{\alpha+i\beta} & = & 0 \text{ for all } \alpha \in \mathrm{Ker}(\phi). \end{array}$$

Therefore

$$\begin{array}{rcl} ||\phi||_g & = & d \max_{i\in\mathbb{Z}}\{\text{there exists } \alpha \in \mathrm{Ker}(\phi) \text{ such that } a_{\alpha+i\beta} \neq 0\} \\ & - & d \min_{i\in\mathbb{Z}}\{\text{there exists } \alpha \in \mathrm{Ker}(\phi) \text{ such that } a_{\alpha+i\beta} \neq 0\} \\ & = & d \max_{i\in\mathbb{Z}}\{\sum_{\alpha\in ker(\phi)} a_{\alpha+i\beta}t^{\alpha+i\beta}\mu^{-i} \neq 0\} \\ & - & d \min_{i\in\mathbb{Z}}\{\sum_{\alpha\in ker(\phi)} a_{\alpha+i\beta}t^{\alpha+i\beta}\mu^{-i} \neq 0\} \\ & = & d \deg(\gamma_\phi(g)). \end{array}$$

□

2.4. **Proof of Proposition 2.1.** We start out with the following basic lemma.

Lemma 2.4. *Let* $f, g \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$*, then* $||\_||_{fg} = ||\_||_f + ||\_||_g$.

This lemma is well–known. It follows from the fact that the Newton polytope of non–commutative multivariable polnyomials fg is the Minkowski sum of the Newton polytopes of f and g.

Lemma 2.5. *Let* $d \in \mathbb{K}(t_1, \dots, t_m)$ *and let* $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ *such that* $d = f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)$*. Then*

$$||\_||_{f_n} - ||\_||_{f_d} = ||\_||_{g_n} - ||\_||_{g_d}.$$

In particular

$$||\_||_d := ||\_||_{f_n} - ||\_||_{f_d}$$

is well–defined.

Proof. Recall that by the definition of the Ore localization $f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)$ is equivalent to the existence of $u, v \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $f_n u = g_n v$ and $f_d u = g_d v$. The lemma now follows immediately from Lemma 2.4. □

Lemma 2.6. *Let* $d, e \in \mathbb{K}(t_1, \dots, t_m)$*, then*

$$||\_||_{de} = ||\_||_d + ||\_||_e.$$

Proof. Pick $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $f_n f_d^{-1} = d$ and $g_n g_d^{-1} = e$. By the Ore property there exist $u, v \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $g_n u = f_d v$. It follows that

$$f_n f_d^{-1} g_n g_d^{-1} = f_n v u^{-1} g_d^{-1} = (f_n v)(g_d u)^{-1}.$$

The lemma now follows immediately from Lemma 2.4. □

We can now give the proof of Proposition 2.1.

Proof of Proposition 2.1. Let B be a matrix defining an element $K_1(\mathbb{K}(t_1,\dots,t_m))$. Assume that we have $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}]$ such that $\det(B) = f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1,\dots,t_m)^\times_{ab}$. We can lift the equality $f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1,\dots,t_m)^\times_{ab}$ to an equality

$$f_n f_d^{-1} = \prod_{i=1}^{r} [a_i, b_i]\, g_n g_d^{-1} \in \mathbb{K}(t_1,\dots,t_m)^\times \tag{1}$$

for some $a_i, b_i \in \mathbb{K}(t_1,\dots,t_m)$. It follows from Lemma 2.6 that $||\_||_{[a_i,b_i]} = 0$. It then follows from Lemma 2.6 that $||\_||_{f_n f_d^{-1}} = ||\_||_{g_n g_d^{-1}}$. □

2.5. **Proof of Theorem 2.2.** Now let $\tau \in K_1(\mathbb{K}(t_1,\dots,t_m))$ which can be represented by a matrix B defined over $\mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}]$. We will show that $||\_||_\tau = ||\_||_B$ defines a seminorm on $\mathrm{Hom}(\mathbb{R}^m,\mathbb{R})$.

Because of the continuity and the $\mathbb{N}$–linearity of $||\_||_B$ it is enough to show that for any two non–trivial homomorphisms $\phi, \tilde{\phi} : \mathbb{Z}^m \to \mathbb{Z}$ we have

$$||\phi + \tilde{\phi}||_B \le ||\phi||_B + ||\tilde{\phi}||_B.$$

Let $\phi, \tilde{\phi} : \mathbb{Z}^m \to \mathbb{Z}$ be non–trivial homomorphisms. Let $d \in \mathbb{Z}$ such that $\mathrm{Im}(\phi) = d\mathbb{Z}$ and pick β with $\phi(\beta) = d$. We write $\mu = t^\beta$. As in Section 2.3 we can form $\mathbb{K}[\mathrm{Ker}(\phi)]$ and we also have an isomorphism $\gamma_\phi : \mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}] \xrightarrow{\cong} (\mathbb{K}[\mathrm{Ker}(\phi)])[s^{\pm1}]$.

Consider $\gamma_\phi(B)$, it is defined over the PID $\mathbb{K}(\mathrm{Ker}(\phi))[s^{\pm1}]$. Therefore we can use elementary row operations to turn $\gamma_\phi(B)$ into a diagonal matrix with entries in $\mathbb{K}(\mathrm{Ker}(\phi))[s^{\pm1}]$. In particular we can find $a_i, b_i \in \mathbb{K}[\mathrm{Ker}(\phi)]$ such that

$$\det(\gamma_\phi(B)) = \sum_{i=r_1}^{r_2} s^i a_i b_i^{-1}$$

Since $\mathbb{K}[\mathrm{Ker}(\phi)]$ is an Ore domain we can in fact find a common denominator for $a_i b_i^{-1}, i = r_1,\dots,r_2$. More precisely, we can find $c_{r_1},\dots,c_{r_2} \in \mathbb{K}[\mathrm{Ker}(\phi)]$ and $d \in \mathbb{K}[\mathrm{Ker}(\phi)]$ such that $a_i b_i^{-1} = c_i d^{-1}$ for $i = r_1,\dots,r_2$. Now let $c = \sum_{i=r_1}^{r_2} s^i c_i$. Then

$$\det(\gamma_\phi(B)) = cd^{-1} \in \mathbb{K}(\mathrm{Ker}(\phi))(s)^\times_{ab}$$

where $c \in \mathbb{K}[\mathrm{Ker}(\phi)][s^{\pm1}]$ and $d \in \mathbb{K}[\mathrm{Ker}(\phi)]$. Now let $f = \gamma_\phi^{-1}(c) \in \mathbb{K}[t_1^{\pm1},\dots,t_m^{\pm1}], g = \gamma_\phi^{-1}(d) \in \mathbb{K}[\mathrm{Ker}(\phi)]$. Then $\det(B) = fg^{-1}$ and by Proposition 2.1 we have

$$||\_||_B = ||\_||_f - ||\_||_g.$$

The crucial observation is that $||\phi||_g = 0$ and $||\phi+\tilde{\phi}||_g = ||\tilde{\phi}||_g$ since $g \in \mathbb{K}[\mathrm{Ker}(\phi)]$. It therefore now follows that

$$\begin{array}{rcl} ||\phi+\tilde{\phi}||_B &=& ||\phi+\tilde{\phi}||_f - ||\phi+\tilde{\phi}||_g \\ &=& ||\phi+\tilde{\phi}||_f - ||\tilde{\phi}||_g \\ &\leq& ||\phi||_f + ||\tilde{\phi}||_f - ||\tilde{\phi}||_g \\ &=& (||\phi||_f - ||\phi||_g) + (||\tilde{\phi}||_f - ||\tilde{\phi}||_g) \\ &=& ||\phi||_B + ||\tilde{\phi}||_B. \end{array}$$

This concludes the proof of Theorem 2.2.

3. Applications to the Thurston norm

3.1. **Reidemeister torsion.** Let X be a finite connected CW–complex. Denote the universal cover of X by $\tilde{X}$. We view $C_*(\tilde{X})$ as a right $\mathbb{Z}[\pi_1(X)]$–module via deck transformations. Let R be a ring. Let $\varphi : \pi_1(X) \to \mathrm{GL}(R, d)$ be a representation, this equips R^d with a left $\mathbb{Z}[\pi_1(X)]$–module structure. We can therefore consider the right R–module chain complex $C_*^{\varphi}(X; R^d) := C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R^d$. We denote its homology by $H_i^{\varphi}(X; R^d)$. If $H_*^{\varphi}(X; R^d) \neq 0$, then we write $\tau(X, \varphi) := 0$. Otherwise we can define the Reidemeister torsion $\tau(X, \varphi) \in K_1(R)/\pm\varphi(\pi_1(X))$. If the homomorphism φ is clear we also write $\tau(X, R^d)$.

Let M be a manifold. Since Reidemeister torsion only depends on the homeomorphism type of the space we can define $\tau(M, \varphi)$ by picking any CW–structure for M. We refer to the excellent book of Turaev [Tu01] for filling in the details.

3.2. **Compatible homomorphisms and the higher order Alexander norm.** In the following let M be a 3–manifold with empty or toroidal boundary, let $\psi : H_1(M) \to \mathbb{Z}^m$ be an epimorphism, and let $\mathbb{K}[t_1^{\pm1}, \dots, t_m^{\pm1}]$ be a multivariable skew Laurent polynomial ring of rank m as in Section 2.1.

A representation $\varphi : \pi_1(M) \to \mathrm{GL}(\mathbb{K}[t_1^{\pm1}, \dots, t_m^{\pm1}], d)$ is called *ψ–compatible* if for any $g \in \pi_1(X)$ we have $\varphi(g) = At^{\psi(g)}$ for some $A \in \mathrm{GL}(\mathbb{K}, d)$. This generalizes definitions in [Tu02b] and [Fr05]. We denote the induced representation $\pi_1(M) \to \mathrm{GL}(\mathbb{K}(t_1, \dots, t_m), d)$ by φ as well and we consider the corresponding Reidemeister torsion $\tau(M, \varphi) \in K_1(\mathbb{K}(t_1, \dots, t_m))/\pm\varphi(\pi_1(M)) \cup \{0\}$.

We say φ is a *commutative representation* if there exists a commutative subfield $\mathbb{F}$ of $\mathbb{K}$ such that for all g we have $\varphi(g) = At^{\psi(g)}$ with A defined over $\mathbb{F}$ and if $t^{\alpha}, t^{\tilde{\alpha}}$ commute for any $\alpha, \tilde{\alpha} \in \mathbb{Z}^m$.

Theorem 3.1. *Let M be a 3–manifold with empty or toroidal boundary. Let $\psi : H_1(M) \to \mathbb{Z}^m$ be an epimorphism. Let $\varphi : \pi_1(M) \to GL(\mathbb{K}[t_1^{\pm1}, \dots, t_m^{\pm1}], d)$ be a ψ–compatible representation such that $\tau(M, \varphi) \neq 0$. If one of the following holds:*

(1) *φ is commutative,*
(2) *there exists $g \in Ker\{\pi_1(M) \to \mathbb{Z}^m\}$ such that $\varphi(g) - id$ is invertible over $\mathbb{K}$,*

then $||\_||_{\tau(M,\varphi)}$ *is a seminorm on* $Hom(\mathbb{R}^m, \mathbb{R})$ *and for any* $\phi : \mathbb{R}^m \to \mathbb{R}$ *we have*

$$||\phi \circ \psi||_T \geq ||\phi||_{\tau(M,\varphi)}.$$

We point out that if $g \in \text{Ker}\{\pi_1(M) \to \mathbb{Z}^m\}$, then $\varphi(g) - \text{id}$ is defined over $\mathbb{K}$ since φ is ψ–compatible. We refer to $||\_||_{\tau(M,\varphi)}$ as the *higher–order Alexander norm.*

In the case that $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ equals $\mathbb{Q}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$, the usual commutative Laurent polynomial ring, we recover McMullen's Alexander norm $||\_||_A$ (cf. [Mc02]). The general commutative case is the main result in [FK05b]. The proof we give here is different in its nature from the proofs in [Mc02] and [FK05b].

Proof. In the case that $m = 1$ it is clear that $||\_||_{\tau(M,\varphi)}$ is a seminorm. The fact that it gives a lower bound on the Thurston norm was shown in [Co04, Ha05, Tu02b, Fr05]. We therefore assume now that $m > 1$.

We first show that $||\phi \circ \psi||_T \geq ||\phi||_{\tau(M,\varphi)}$ for any $\phi : \mathbb{R}^m \to \mathbb{R}$. Since both sides are $\mathbb{N}$–linear and continuous we only have to show that $||\phi \circ \psi||_T \geq ||\phi||_{\tau(M,\varphi)}$ for all epimorphisms $\phi : \mathbb{Z}^m \to \mathbb{Z}$. So let $\phi : \mathbb{Z}^m \to \mathbb{Z}$ be an epimorphism.

Pick $\mu \in \mathbb{Z}^m$ with $\phi(\mu) = 1$ as in the definition of $\deg_\phi(\tau(M, \varphi))$. We can then again form the rings $\mathbb{K}[\text{Ker}(\phi)][s^{\pm 1}]$ and $\mathbb{K}(\text{Ker}(\phi))(s)$. First note that by Theorem 2.3

$$||\phi||_{\tau(M,\varphi)} = \deg_\phi(\tau(M, \varphi))$$

since ϕ is surjective. The representation

$$\pi_1(M) \to \text{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d) \to \text{GL}(\mathbb{K}(\text{Ker}(\phi))[s^{\pm 1}], d)$$

is ϕ–compatible since $\pi_1(M) \to \text{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ is ψ–compatible. It now follows from [Fr05, Theorem 1.2] that $||\phi \circ \psi||_T \geq \deg(\tau(M, \mathbb{K}(\text{Ker}(\phi))(s))) = \deg_\phi(\tau(M, \varphi))$ (cf. also [Tu02b]).

In the remainder of the proof we will show that if $m > 1$ then the Reidemeister torsion $\tau(M, \varphi) \in K_1(\mathbb{K}(t_1, \dots, t_m))/ \pm \varphi(\pi_1(M))$ can be represented by a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. It then follows from Theorem 2.2 that $||\_||_{\tau(M,\varphi)}$ is a seminorm.

First consider the case that φ is a commutative representation. Let $\mathbb{F}$ be the commutative subfield $\mathbb{F}$ in the definition of a commutative representation. Denote by $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ the ordinary Laurent polynomial ring. Then we have ψ–compatible representations $\pi_1(M) \to \text{GL}(\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d) \hookrightarrow \text{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$. By [Tu01, Proposition 3.6] we have

$$\tau(M, \mathbb{F}(t_1, \dots, t_m)) = \tau(M, \mathbb{K}(t_1, \dots, t_m)) \in K_1(\mathbb{K}(t_1, \dots, t_m))/ \pm \varphi(\pi_1(M)).$$

Since $m > 1$ it follows from [Tu01, Theorem 4.7] combined with [FK05b, Lemmas 6.2 and 6.5] that $\det(\tau(M, \mathbb{F}(t_1, \dots, t_m))) \in \mathbb{F}(t_1, \dots, t_m)$ equals the twisted multivariable Alexander polynomial, in particular it is defined over $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. This concludes the proof in the commutative case.

It therefore remains to consider the case that there exists $g \in \text{Ker}\{G \to \mathbb{Z}^m\}$ such that $\varphi(g) - \text{id}$ is invertible. We first consider the case that M is a closed 3–manifold.

Let $h = g$. Now pick a Heegard decomposition $M = G_0 \cup H_0$. We can add a handle to G_0 in $M \setminus G_0$ so that the core represents g. Adding further handles in $M \setminus G_0$ we can assume that the complement is again a handlebody. We call the two handlebodies G_1 and H_1.

Now we can add a handle to H_1 in $M \setminus G_1$ so that the core represents h. Adding further handles in $M \setminus H_1$ we can assume that the complement is again a handlebody. We call the two handlebodies G and H. Note that g is still represented by a handle of G. Now give M the CW structure as follows: Take one 0–cell, attach 1–cells along a choice of cores of G such that g corresponds to one 1–cell. Attach 2–cells along cocores of H such that one cocore corresponds to h. Finally attach one 3–cell.

Denote the number of 1–cells by n. Consider the chain complex of the universal cover $\tilde{M}$:

$$0 \to C_3(\tilde{M})^1 \xrightarrow{\partial_3} C_2(\tilde{M})^n \xrightarrow{\partial_2} C_1(\tilde{M})^n \xrightarrow{\partial_1} C_0(\tilde{M})^1 \to 0,$$

where the supscript indicates the rank over $\mathbb{Z}[\pi_1(M)]$. Picking appropriate lifts of the cells of M to cells of $\tilde{M}$ and picking an appropriate order we get bases for the $\mathbb{Z}[\pi_1(M)]$–modules $C_i(\tilde{M})$, such that if A_i denotes the matrix corresponding to ∂_i, then A_1 and A_3 are of the form

$$\begin{aligned} A_3 &= (1-g, 1-g_2, \dots, 1-g_n)^t, \\ A_1 &= (1-h, 1-h_2, \dots, 1-h_n), \end{aligned}$$

for some $g_i, h_i \in \pi_1(M), i = 2, \dots, n$. By assumption $\mathrm{id} - \varphi(g)$ and $\mathrm{id} - \varphi(h)$ are invertible over $\mathbb{K}$. Denote by B_2 the result of deleting the first column and the first row of A_2. Let $\tau := (\mathrm{id} - \varphi(g))^{-1}\varphi(B_2)(\mathrm{id} - \varphi(h))^{-1}$. Note that τ is defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Since we assume that $\tau(M, \varphi) \neq 0$ it follows that $\varphi(B_2)$ is invertible over $\mathbb{K}(t_1, \dots, t_m)$ and $\tau(M, \varphi) = \tau \in K_1(\mathbb{K}(t_1, \dots, t_m))/\pm\varphi(\pi_1(M))$ (we refer to [Tu01, Theorem 2.2] for details). Therefore $\tau(M, \varphi) \in K_1(\mathbb{K}(t_1, \dots, t_m))/\pm\varphi(\pi_1(M))$ can be represented by a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

In the case that M is a 3–manifold with non–empty toroidal boundary we can find a (simple) homotopy equivalence to a 2–complex X with $\chi(X) = 0$. We can assume that the CW–structure has one 0–cell, n 1–cells and $n-1$ 2–cells, furthermore we can assume that one of the 1–cells represents an element $h \in \mathrm{Ker}\{\psi : G \to \mathbb{Z}^m\}$ such that $\mathrm{id} - \varphi(h)$ is invertible. We get a chain complex

$$0 \to C_2(\tilde{X})^{n-1} \xrightarrow{\partial_2} C_1(\tilde{X})^n \xrightarrow{\partial_1} C_0(\tilde{X})^1 \to 0.$$

Picking appropriate lifts of the cells of X to cells of $\tilde{X}$ we get bases for the $\mathbb{Z}[\pi_1(X)]$–modules $C_i(\tilde{X})$, such that if A_i denotes the matrix corresponding to ∂_i, then A_1 is of the form

$$A_1 = (1-h, 1-h_2, \dots, 1-h_n), h_i \in \pi_1(M).$$

Now denote by B_2 the result of deleting the first row of A_2. Then $\tau := \varphi(B_2)(\mathrm{id} - \varphi(h))^{-1}$ is again defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and the proof continues as in the case of a closed 3–manifold. □

Remark. Note that if follows from [Fr05] that if M is closed, or if M has toroidal boundary, then $\tau(M,\varphi) \neq 0$ is equivalent to $H_1(M;\mathbb{K}(t_1,\dots,t_m)) = 0$, or equivalently, that $H_1(M;\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}])$ has rank zero over $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$.

Remark. Note that the computation of $f_d \in \mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ and $f_n \in \mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ such that $\det(\tau(M,\varphi)) = f_n f_d^{-1}$ is computationally equivalent to the computation of $\deg_\phi(\tau(M,\varphi))$ for some $\phi : H_1(M) \to \mathbb{Z}$. Put differently we get the perhaps surprising fact that computing the higher–order Alexander norm does not take longer than computing a single higher–order one–variable Alexander polynomial.

4. Examples of ψ–compatible homomorphisms

4.1. **Skew fields of group rings.** A group G is called locally indicable if for every finitely generated subgroup $U \subset G$ there exists a non–trivial homomorphism $U \to \mathbb{Z}$.

Theorem 4.1. *Let G be a locally indicable and amenable group and let R be a subring of $\mathbb{C}$. Then $R[G]$ is an Ore domain, in particular it embeds in its classical right ring of quotients $\mathbb{K}(G)$.*

It follows from [Hi40] that $R[G]$ has no zero divisors. The theorem now follows from [Ta57] or [DLMSY03, Corollary 6.3].

A group G is called poly–torsion–free–abelian (PTFA) if there exists a filtration

$$1 = G_0 \subset G_1 \subset \dots \subset G_{n-1} \subset G_n = G$$

such that G_i/G_{i-1} is torsion free abelian. It is well–known that PTFA groups are amenable and locally indicable (cf. [St74]). The group rings of PTFA groups played an important role in [COT03], [Co04] and [Ha05].

4.2. **Admissible pairs and multivariable skew Laurent polynomial rings.** We slightly generalize a definition from [Ha06].

Definition. Let π be a group and let $\psi : \pi \to \mathbb{Z}^m$ be an epimorphism and let $\varphi : \pi \to G$ be an epimorphism to a locally indicable and amenable group G such that there exists a map $G \to \mathbb{Z}^m$ (which we also denote by ψ) such that

$$\begin{array}{ccc} \pi & \xrightarrow{\varphi} & G \\ & {\scriptstyle\psi}\searrow & \downarrow{\scriptstyle\psi} \\ & & \mathbb{Z}^m \end{array}$$

commutes. Following [Ha06, Definition 1.4] we call (φ,ψ) an *admissible pair* for π.

Clearly $G_\psi := \mathrm{Ker}\{G \to \mathbb{Z}^m\}$ is locally indicable and amenable. It follows now from [Pa85, Lemma 3.5 (ii), p. 609] that $(\mathbb{Z}[G], \mathbb{Z}[G_\psi]\setminus\{0\})$ satisfies the Ore property. Now pick elements $t^\alpha \in G, \alpha \in \mathbb{Z}^m$ such that $\psi(t^\alpha) = \alpha$ and $t^{n\alpha} = (t^\alpha)^n$ for any $\alpha \in \mathbb{Z}^m, n \in \mathbb{Z}$.

Clearly $\mathbb{Z}[G](\mathbb{Z}[G_\psi]\setminus\{0\})^{-1}=\sum_{\alpha\in\mathbb{Z}^m}\mathbb{K}(G_\psi)t^\alpha$ is a multivariable skew Laurent polynomial ring of rank m over the field $\mathbb{K}(G_\psi)$ as defined in Section 2.1. We denote this ring by $\mathbb{K}(G_\psi)[t_1^{\pm1},\dots,t_m^{\pm1}]$. Note that $\mathbb{Z}[\pi]\to\mathbb{Z}[G]\to\mathbb{K}(G_\psi)[t_1^{\pm1},\dots,t_m^{\pm1}]$ is a ψ–compatible homomorphism and that $\mathbb{K}(G_\psi)(t_1,\dots,t_m)$ is canonically isomorphic to $\mathbb{K}(G)$.

A family of examples of admissible pairs is provided by the rational derived series of a group π introduced by the second author (cf. [Ha05, Section 3]). Let $\pi_r^{(0)}:=\pi$ and define inductively

$$\pi_r^{(n)}:=\left\{g\in\pi_r^{(n-1)}|\,g^d\in\left[\pi_r^{(n-1)},\pi_r^{(n-1)}\right]\text{ for some }d\in\mathbb{Z}\setminus\{0\}\right\}.$$

Note that $\pi_r^{(n-1)}/\pi_r^{(n)}\cong\left(\pi_r^{(n-1)}/\left[\pi_r^{(n-1)},\pi_r^{(n-1)}\right]\right)/\mathbb{Z}$–torsion. By [Ha05, Corollary 3.6] the quotients $\pi/\pi_r^{(n)}$ are PTFA groups for any π and any n. If $\psi:\pi\to\mathbb{Z}^m$ is an epimorphism, then $(\pi\to\pi/\pi_r^{(n)},\psi)$ is an admissible pair for π) for any $n>0$.

4.3. **Admissible pairs and seminorms.** Let M be a 3–manifold with empty or toroidal boundary. Let $(\varphi:\pi_1(M)\to G,\psi:\pi_1(M)\to\mathbb{Z}^m)$ be an admissible pair for $\pi_1(M)$. We denote the induced map $\mathbb{Z}[\pi_1(M)]\to\mathbb{K}(G_\psi)(t_1,\dots,t_m)$ by φ as well.

Let $\phi:\mathbb{Z}^m\to\mathbb{Z}$ be a non–trivial homomorphism. We denote the induced homomorphism $G\to\mathbb{Z}^m\to\mathbb{Z}$ by ϕ as well. We write $G_\phi:=\mathrm{Ker}\{G\to\mathbb{Z}\}$. Pick $\mu\in G$ such that $\phi(\mu)\mathbb{Z}=\mathrm{Im}(\phi)$. We define $\mathbb{Z}[G_\phi][u^{\pm1}]$ via $uf=\mu f\mu^{-1}u$. Note that we get an isomorphism $\mathbb{K}(G_\phi)(u)\cong\mathbb{K}(G)$. If $\tau(M,\varphi)\neq0$, then we define

$$\delta_G(\phi):=\max\{0,\deg(\tau(M,\mathbb{K}(G_\phi)(u)))\}$$

otherwise we write $\delta_G(\phi)=-\infty$. We will adopt the convention that $-\infty<a$ for any $a\in\mathbb{Z}$. By [Fr05] this agrees with the definition in [Ha06, Definition 1.6] if $\delta_G(\phi)\neq-\infty$ and if $\varphi:G\to\mathbb{Z}^m$ is not an isomorphism or if $m>1$. In the case that $\varphi:G\to\mathbb{Z}$ is an isomorphism and $M\neq S^1\times D^2,S^1\times S^2$, this definition differs from [Ha06, Definition 1.6] by the term $1+b_3(M)$. In the case that $\varphi:\pi\to\pi/\pi_r^{(n+1)}$ then we also write $\delta_n(\phi)=\delta_{\pi/\pi_r^{(n+1)}}(\phi)$.

Theorem 4.2. *Let M be a 3–manifold with empty or toroidal boundary. Let $(\varphi:\pi_1(M)\to G,\psi:\pi_1(M)\to\mathbb{Z}^m)$ be an admissible pair for $\pi_1(M)$ such that $\tau(M,\varphi)\neq0$. Then for any $\phi:\mathbb{Z}^m\to\mathbb{Z}$ we have*

$$||\phi||_{\tau(M,\varphi)}=\delta_G(\phi),$$

and $\phi\mapsto\max\{0,\delta_G(\phi)\}$ defines a seminorm which is a lower bound on the Thurston norm.

Note that this theorem implies in particular Theorem 1.1.

Proof. Let $\phi:\mathbb{Z}^m\to\mathbb{Z}$ be a non–trivial homomorphism. As in Section 2.1 we can form $\mathbb{K}(G_\phi)[s^{\pm1}]$ and $\mathbb{K}(G_\psi)(\mathrm{Ker}(\phi))[s^{\pm1}]$. Note that these rings are canonically isomorphic Laurent polynomial rings. If $\psi:G\to\mathbb{Z}^m$ is an isomorphism, then φ is commutative.

Otherwise we can find a non–trivial $g \in \text{Ker}(\psi)$, so clearly $1 - \varphi(g) \neq 0 \in \mathbb{K}(G)$. This shows that we can apply Theorem 3.1 which then concludes the proof. $\square$

In the case that $\varphi : \pi \to \pi/\pi_r^{(n+1)}$ we denote the seminorm $\phi \mapsto \max\{0, \delta_n(\phi)\}$ by $||\_||_n$. Note that in the case $n = 0$ this was shown by the second author [Ha05, Proposition 5.12] to be equal to McMullen's Alexander norm [Mc02].

4.4. **Admissible triple.** We now slightly extend a definition from [Ha06].

Definition. Let π be a group and $\psi : \pi \to \mathbb{Z}^m$ an epimorphism. Furthermore let $\varphi_1 : \pi \to G_1$ and $\varphi_2 : \pi \to G_2$ be epimorphisms to locally indicable and amenable groups G_1 and G_2. We call $(\varphi_1, \varphi_2, \psi)$ an *admissible triple* for π if there exist epimorphisms $\Phi : G_1 \to G_2$ and $\psi_2 : G_2 \to \mathbb{Z}^m$ such that $\varphi_2 = \Phi \circ \varphi_1$, and $\psi = \psi_2 \circ \varphi_2$.

Note that in particular $(\varphi_i, \psi), i = 1, 2$ are admissible pairs for π. Combining Theorem 4.2 with [Fr05, Theorem 1.3] (cf. also [Ha06]) we get the following result.

Theorem 4.3. *Let M be a 3–manifold with empty or toroidal boundary. If $(\varphi_1, \varphi_2, \psi)$ is an admissible triple for $\pi_1(M)$ such that $\tau(M, \varphi_2) \neq 0$, then we have the following inequalities of seminorms:*

$$||\_||_{\tau(M,\varphi_2)} \leq ||\_||_{\tau(M,\varphi_1)} \leq ||\_||_T.$$

In particular we have

$$||\_||_0 \leq ||\_||_1 \leq \cdots \leq ||\_||_T.$$

Let M be a 3–manifold with empty or toroidal boundary and let $\phi \in H^1(M; \mathbb{Z})$. Since $\delta_n(\phi) \in \mathbb{N}$ for all n it follows immediately from Theorem 4.3 that there exists $N \in \mathbb{N}$ such that $\delta_n(\phi) = \delta_N(\phi)$ for all $n \geq N$. But we can in fact prove a slightly stronger statement, namely that there exists such an N independent of the choice of $\phi \in H^1(M; \mathbb{Z})$.

Proposition 4.4. *Let M be a 3–manifold with empty or toroidal boundary. There exists $N \in \mathbb{N}$ such that $\delta_n(\phi) = \delta_N(\phi)$ for all $n \geq N$ and all $\phi \in H^1(M; \mathbb{R})$.*

Proof. Write $\pi = \pi_1(M), \pi_n = \pi/\pi_r^{(n+1)}$ and $m = b_1(M)$. Let $\psi : \pi \to \mathbb{Z}^m$ be an epimorphism. Write $(\pi_n)_\psi = \text{Ker}\{\psi : \pi_n \to \mathbb{Z}^m\}$. Now pick elements $t^\alpha \in \pi_n, \alpha \in \mathbb{Z}^m$ such that $\psi(t^\alpha) = \alpha$ and $t^{k\alpha} = (t^\alpha)^k$ for any $\alpha \in \mathbb{Z}^m, k \in \mathbb{Z}$. Consider the map $\mathbb{Z}[\pi] \to \mathbb{Z}[\pi_n] \to \mathbb{K}((\pi_n)_\psi)(t_1, \dots, t_m)$. We write $\tau_n = \tau(M, \mathbb{K}((\pi_n)_\psi)(t_1, \dots, t_m))$. We can find $f_n, g_n \in \mathbb{K}((\pi_n)_\psi) \in [t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $\tau_n = f_n g_n^{-1}$.

Given a seminorm s on $H^1(N; \mathbb{R})$ whose normball is a (possibly non–compact) polygon we can study its dual polytope $d(s)$. Note that given $f = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha \in \mathbb{K}((\pi_n)_\psi) \in [t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ the dual polytope $d(||\_||_f)$ equals the Newton polygon $N(f)$ which is the convex hull of $\{\alpha | a_\alpha \neq 0\}$. Clearly $d(||\_||_f)$ has only integral vertices.

By the definition of $\delta_n = ||\_||_{\tau_n} = ||\_||_{f_g g_n^{-1}}$ it follows that

$$d(\delta_n) + d(g_n) = d(\tau_n) + d(g_n) = d(f_n)$$

where "+" denotes the Minkowski sum of convex sets. It is easy to see that this implies that $d(\delta_n)$ has only integral vertices.

Theorem 4.3 implies that there is a sequence of inclusions

$$d(\delta_0) \subset d(\delta_1) \subset \cdots \subset d(||\_||_T).$$

Since $d(||\_||_T)$ is compact and since $d(\delta_n)$ has integral vertices for all n it follows immediately that there exists $N \in \mathbb{N}$ such that $d(\delta_n) = d(\delta_N)$ for all $n \geq N$. This completes the proof of the proposition. □

5. Examples

Before we discuss the Thurston norm of a family of links we first need to introduce some notation for knots. Let K be a knot. We denote the knot complement by $X(K)$. Let $\phi : H_1(X(K)) \to \mathbb{Z}$ be an isomorphism. We write $\delta_n(K) := \delta_n(\phi)$. This agree with the original definition of Cochran [Co04] for $n > 0$ and if $\Delta_K(t) = 1$, and it is one less than Cochran's definition otherwise.

In the following let $L = L_1 \cup \cdots \cup L_m$ be any ordered oriented m–component link. Let $i \in \{1, \ldots, m\}$. Let K be an oriented knot with $\Delta_K(t) \neq 1$ which is separated from L by a sphere S. We pick a path from a point on K to a point on L_i and denote by $L\#_i K$ the link given by performing the connected sum of L_i with K (cf. Figure 1). Note that this connected sum is well–defined, i.e. independent of the choice of the path. We will study the Thurston norm of $L\#_i K$.

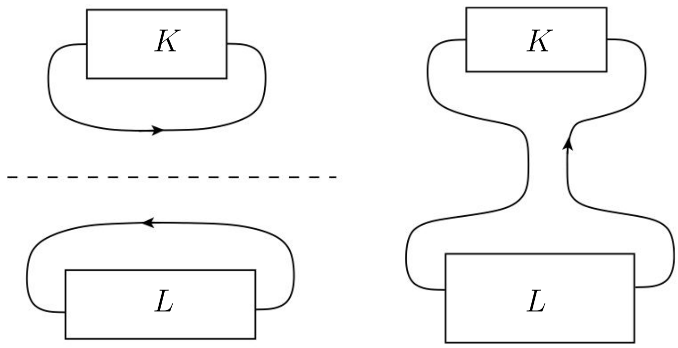

Figure 1. The link $L\#_i K$.

Now assume that L is a non–split link with at least two components and such that $||\_||_0 = ||\_||_T$. Many examples of such links are known (cf. [Mc02]). For the link $L\#_i K$ denote its meridians by $\mu_i, i = 1, \ldots, m$. Let $\psi : H_1(X(L\#_i K)) \to \mathbb{Z}^m$ be the isomorphism given by $\psi(\mu_i) = e_i$, where e_i is the i–th vector of the standard basis of $\mathbb{Z}^m$.

We write $\pi := \pi_1(X(L\#_i K))$. For all $\alpha \in \mathbb{Z}^m$ we pick $t^\alpha \in \pi/\pi_r^{(n+1)}$ with $\psi(t^\alpha) = \alpha$ and such that $t^{l\alpha} = (t^\alpha)^l$ for all $\alpha \in \mathbb{Z}^m$ and $l \in \mathbb{Z}$. Furthermore write $t_i := t^{e_i}$.

Proposition 5.1. *Consider the natural map*

$$\varphi : \pi \to \mathbb{K}(\pi/\pi_r^{(n+1)}) = \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_1, \dots, t_m).$$

where π is as defined above. There exists an element $f(t_i) \in \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_i^{\pm 1}] \subset \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $deg(f(t_i)) = \delta_n(K) + 1$, and there exists a $d = d(t_1, \dots, t_m) \in \mathbb{K}(t_1, \dots, t_m)$ with $||\_||_d = ||\_||_0$, such that

$$(2) \quad \tau(X(L\#_i K), \varphi) = d(t_1, \dots, t_m) f(t_i) \in K_1(\mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_1, \dots, t_m))/ \pm \varphi(\pi).$$

Furthermore, if $\delta_n(K) = 2genus(K) - 1$, then

$$||\_||_{\tau(X(L\#_i K),\varphi)} = ||\_||_T.$$

Proof. Let S be the embedded sphere in S^3 coming from the definition of the connected sum operation (cf. Figure 1). Let D be the annulus $S \cap X(L\#_i K)$ and we denote by P the closure of the component of $X(L\#_i K) \setminus D$ corresponding to K. We denote the closure of the other component by P' (see Figure 2 below). Note that P is homeomorphic to $X(K)$ and P' is homeomorphic to $X(L)$. Denote the induced maps

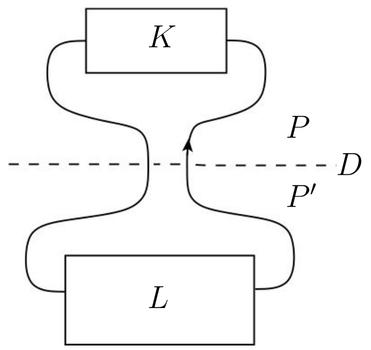

FIGURE 2. The link complement of $L\#_i K$ cut along the annulus D.

to $(K) := \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_1, \dots, t_m)$ by φ as well. We get an exact sequence

$$0 \to C_*^\varphi(D; (K)) \to C_*^\varphi(P; (K)) \oplus C_*^\varphi(P'; (K)) \to C_*^\varphi(X(L\#_i K); (K)) \to 0$$

of chain complexes. It follows from [Tu01, Theorem 3.4] that

$$(3) \qquad \tau(P, \varphi)\tau(P', \varphi) = \tau(D, \varphi)\tau(X(L_i\#K), \varphi) \in \big(K_1((K))/ \pm \varphi(\pi)\big) \cup \{0\}.$$

First note that D is homotopy equivalent to a circle and that $\mathrm{Im}\{\psi : \pi_1(D) \to \mathbb{Z}^m\} = \mathbb{Z}e_i$. It is now easy to see that $\tau(D, \varphi) = (1 - at_i)^{-1}$ for some $a \in \mathbb{K}(\pi_\psi/\pi_r^{(n+1)}) \setminus \{0\}$.

Next note that $\mathrm{Im}\{\psi : \pi_1(P) \to \mathbb{Z}^m\} = \mathbb{Z}e_i$. In particular $\tau(P, \varphi)$ is defined over the one–variable Laurent polynomial ring $\mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_i^{\pm 1}]$ which is a PID. Recall that we can therefore assume that its Dieudonné determinant $f(t_i)$ lies in $\mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_i^{\pm 1}]$ as well.

Claim.
$$\deg(\tau(P, \varphi : \pi_1(P) \to \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_i)) = \delta_n(K).$$

First recall that there exists a homeomorphism $P \cong X(K)$. We also have an inclusion $X(L\#_i K) \to X(L_i\#K)$. Combining with the degree one map $X(L_i\#K) \to X(K)$ we get a factorization of an automorphism of $\pi_1(X(K))$ as follows:
$$\pi_1(X(K)) \cong \pi_1(P) \to \pi_1(X(L\#_i K)) \to \pi_1(X(L_i\#K)) \to \pi_1(X(K)).$$
Since the rational derived series is functorial (cf. [Ha05]) we in fact get that
$$\begin{array}{rcl} \pi_1(X(K))/\pi_1(X(K))_r^{(n+1)} & \cong & \pi_1(P)/\pi_1(P)_r^{(n+1)} \\ & \to & \pi_1(X(L_i\#K))/\pi_1(X(L_i\#K))_r^{(n+1)} \\ & \to & \pi_1(X(K))/\pi_1(X(K))_r^{(n+1)} \end{array}$$
is an isomorphism. In particular
$$\pi_1(X(K))/\pi_1(X(K))_r^{(n+1)} \to \pi_1(X(L\#_i K))/\pi_1(X(L\#_i K))_r^{(n+1)}$$
is injective, and the induced map on Ore localizations is injective as well. Finally note that $\mathrm{Ker}\{\pi_1(X(K)) \to \pi_1(P) \xrightarrow{\psi} \mathbb{Z}^m\} = \mathrm{Ker}(\phi)$ where $\phi : \pi_1(X(K)) \to \mathbb{Z}$ is the abelianization map. It now follows that
$$\begin{array}{rcl} \delta_n(K) & = & \deg(\tau(X(K), \pi_1(X(K)) \to \mathbb{K}(\pi_1(X(K))_\phi/\pi_1(X(K))_r^{(n+1)})(t_i)) \\ & = & \deg(\tau(X(K), \pi_1(X(K)) \to \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_i)) \\ & = & \deg(\tau(P, \pi_1(P) \to \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_i)). \end{array}$$
Note that the second equality follows from the functoriality of torsion (cf. [Tu01, Proposition 3.6]) and the fact that going to a supfield does not change the degree of a rational function. This concludes the proof of the claim.

Claim. We have the following equality of norms on $H^1(X(L);\mathbb{Z})$:
$$||{-}||_{\tau(P',\varphi)} = ||{-}||_T.$$

First recall that P' is homeomorphic to $X(L)$. The claim now follows immediately from Theorem 4.3 applied to φ and to the abelianization map of $\pi_1(P')$, and from the assumption that $||{-}||_0 = ||{-}||_T$ on $H^1(X(L);\mathbb{Z})$.

Putting these computations together and using Equation (3) we now get a proof of Equation (2).

Now assume that $\delta_n(K) = 2\mathrm{genus}(K) - 1$. Let S_i be a Seifert surface of K with minimal genus. Let $\phi : \mathbb{Z}^m \to \mathbb{Z}$ be an epimorphism and let $l = \phi(\mu_i) \in \mathbb{Z}$. We first view ϕ as an element in $\mathrm{Hom}(H_1(X(L);\mathbb{Z})$. A standard argument shows that ϕ is dual to a (possibly disconnected) surface S which intersects the tubular neighborhood of L_i in exactly l disjoint curves. Then the connected sum S' of S with l copies of S_i gives a surface in $X(L\#_i K)$ which is dual to ϕ viewed as an element in $\mathrm{Hom}(H_1(X(L\#_i K);\mathbb{Z})$. A standard argument shows that S' is Thurston norm minimizing (cf. e.g. [Lic97, p. 18]).

Clearly $\chi(S') = \chi(S) + l(\chi(S_i) - 1)$. A straightforward argument shows that furthermore $\chi_-(S') = \chi_-(S) + l(\chi_-(S_i) + 1)$ since L is not a split link and since K is non–trivial.

We now compute

$$\begin{aligned} ||\phi||_T &= \chi_-(S') \\ &= \chi_-(S) - n(\chi(S_i) - 1) \\ &= ||\phi||_T + 2l\text{genus}(K) \\ &= ||\phi||_d + 2(\delta_n(K) + 1) \\ &= ||\phi||_d + 2\deg(f(t_i)) \\ &= ||\phi||_{\tau(X(L\#_i K),\varphi)}. \end{aligned}$$

By the $\mathbb{R}$–linearity and the continuity of the norms it follows that

$$||\phi||_{\tau(X(L\#_i K),\varphi)} = ||\phi||_T$$

for all $\phi : \mathbb{Z}^m \to \mathbb{R}$. □

Denote by $\Diamond(n, m)$ the convex polytope given by the vertices $(\pm\frac{1}{n}, 0)$ and $(0, \pm\frac{1}{m})$. Let $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ be never decreasing sequences of odd positive numbers which are eventually constant, i.e. there exists an N such that $n_i = n_N$ for all $i \geq N$ and $m_i = m_N$ for all $i \geq N$. According to [Co04] we can find knots K_1 and K_2 such that $\delta_i(K_1) = n_i$ for any i, $\delta_N(K_1) = 2\,\text{genus}(K_1) - 1$ and $\delta_i(K_2) = m_i$ for any i and $\delta_N(K_2) = 2\,\text{genus}(K_2) - 1$.

Let $H(K_1, K_2)$ be the link formed by adding the two knots K_1 and K_2 from above to the Hopf link (cf. Figure 3). Recall that the Thurston norm ball of the Hopf link is given by $\Diamond(1, 1)$. Let $\pi := \pi_1(X(L))$. It follows immediately from applying

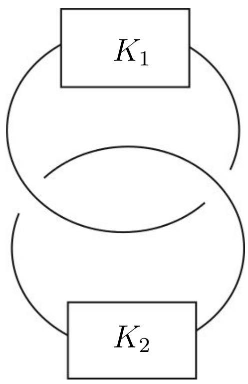

FIGURE 3. $H(K_1, K_2)$ is obtained by tying K_1 and K_2 into the Hopf link

Proposition 5.1 twice that the norm ball of $||\_||_i$ equals $\Diamond(n_i + 1, m_i + 1)$ and that $||\_||_N = ||\_||_T$. The following result is now an immediate consequence of Proposition 5.1.

Corollary 5.2. *We have the following sequence of inequalities of seminorms*

$$||\_||_A = ||\_||_0 \le ||\_||_1 \le ||\_||_2 \le \dots \le ||\_||_N = ||\_||_T.$$

In [Ha05] the second author gave examples of 3–manifolds M such that

$$||\_||_A = ||\_||_0 \le ||\_||_1 \le ||\_||_2 \le \dots$$

but in that case it was not known whether the sequence of norms $||\_||_i$ eventually agrees with $||\_||_T$.

It is an interesting question to determine which 3–manifolds satisfy $||\_||_T = ||\_||_n$ for large enough n. We conclude this paper with the following conjecture.

Conjecture 5.3. *If* $\pi_1(M)_r^{(\omega)} \equiv \bigcap_{n\in\mathbb{N}} \pi_1(M)_r^{(n)} = \{1\}$, *then there exists* $n \in \mathbb{N}$ *such that* $||\_||_T = ||\_||_n$.

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