

# FULL SIGNATURE INVARIANTS FOR $L_0(F(t))$

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ABSTRACT. Let  $F/\mathbb{Q}$  be a number field closed under complex conjugation. Denote by  $L_0(F(t))$  the Witt group of hermitian forms over  $F(t)$ . We find full invariants for detecting non-zero elements in  $L_0(F(t)) \otimes \mathbb{Q}$ , this group plays an important role in topology in the work done by Casson and Gordon.

## 1. $L$ -GROUPS AND SIGNATURES

Let  $R$  be a ring with (possibly trivial) involution. An  $\epsilon$ -hermitian ( $\epsilon = \pm 1$ ) form is a sesquilinear map  $\theta : V \times V \rightarrow R$  over a finitely generated free  $R$ -module with the properties that  $\theta(ra, b) = \bar{r}\theta(a, b)$ ,  $\theta(a, rb) = \theta(a, b)r$  and  $\theta(a, b) = \epsilon\theta(b, a)$  for all  $a, b \in V, r \in R$ . It is called non-singular if the map  $V \rightarrow \text{Hom}(V, R), a \mapsto (b \mapsto \theta(b, a))$  is an isomorphism. We denote by  $L_0(R, \epsilon), \epsilon = \pm 1$ , the Witt group of  $\epsilon$ -hermitian non-singular forms (cf. [L93] and [R98]). More precisely denote by  $M$  the groupoid under direct sum of  $\epsilon$ -hermitian non-singular forms. Let  $\sim$  be the equivalence relation generated by setting any form  $(V, \theta)$  to zero which has a submodule of half-rank on which  $\theta$  vanishes. Then define  $L_0(R, \epsilon) := M / \sim$ , this is a group (cf. [L93]) under the direct sum operation. We'll abbreviate  $L_0(R)$  for  $L_0(R, +1)$ .

Let  $F \subset \mathbb{C}$  be a subfield, closed under complex conjugation. We will always equip the rings  $F[t, t^{-1}], F(t)$  with the involution given by the complex involution on  $F$  and  $\bar{t} := t^{-1}$ . For  $\tau = [(V, \theta)] \in L_0(F, \epsilon)$  we define

$$\text{sign}(\tau) := \dim(V^+) - \dim(V^-)$$

where  $V^+$  (resp.  $V^-$ ) denotes the maximal positive (resp. negative) subspace of  $\sqrt{\epsilon}\theta$ . This number is independent of the choice of representative  $(V, \theta)$ .

If  $F \subset \mathbb{C}$  is a subfield such that all positive elements are squares, then by Sylvester's theorem

$$\begin{aligned} \text{sign} : L_0(F, \epsilon) &\rightarrow \mathbb{Z} \\ (V, \theta) &\mapsto \text{sign}(V, \theta) \end{aligned}$$

is an isomorphism. In particular  $L_0(\mathbb{C}, \pm 1) = L_0(\bar{\mathbb{Q}}, \pm 1) \cong \mathbb{Z}$  via the signature map, where we denote by  $\bar{\mathbb{Q}} \subset \mathbb{C}$  the algebraic closure of  $\mathbb{Q}$ . Since we are interested in studying to which degree signatures determine forms we'll work in this paper with  $\tilde{L}_0(R, \epsilon) := L_0(R, \epsilon) \otimes \mathbb{Q}$ , i.e. we ignore the torsion part of  $L_0(R, \epsilon)$ . Note that the above maps extend to an isomorphism  $\text{sign} : L_0(F, \epsilon) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ .

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Let  $F$  be a Galois extension of  $\mathbb{Q}$  with a (possibly trivial) involution. Denote by  $G(F)$  the set of all  $\mathbb{Q}$ -linear embeddings  $F \rightarrow \bar{\mathbb{Q}}$  preserving the involution. For  $\rho \in G(F)$  denote the induced maps  $\tilde{L}_0(F, \epsilon) \rightarrow \tilde{L}_0(\bar{\mathbb{Q}}, \epsilon)$ ,  $\tilde{L}_0(F(t), \epsilon) \rightarrow \tilde{L}_0(\bar{\mathbb{Q}}(t), \epsilon), \dots$  by  $\rho$  as well. Denote by  $G_0(F) \subset G(F)$  any subset such that for each  $\rho \in G(F)$  there exists precisely one  $\tilde{\rho} \in G_0(F)$  with  $\tilde{\rho} = \rho$  or  $\tilde{\rho} = \bar{\rho}$ .

Let  $\tau = [(V, \theta)] \otimes \frac{p}{q} \in \tilde{L}_0(\bar{\mathbb{Q}}(t), \epsilon)$ . If  $z \in S^1$  is transcendental then we consider  $\mathbb{C}$  as a  $\bar{\mathbb{Q}}(t)$  module via  $f(t) \cdot w := f(z)w$ . Denote the pairing

$$\begin{aligned} V \otimes_{\bar{\mathbb{Q}}(t)} \mathbb{C} \times V \otimes_{\bar{\mathbb{Q}}(t)} \mathbb{C} &\rightarrow \mathbb{C} \\ (f_1(t) \otimes z_1, f_2(t) \otimes z_2) &\mapsto \bar{z}_1 \theta(f_1(t), f_2(t))|_{t=zz_2} \end{aligned}$$

by  $\theta$  as well. Then  $\tau(z) := [(V \otimes_{\bar{\mathbb{Q}}(t)} \mathbb{C}, \theta)] \otimes \frac{p}{q} \in \tilde{L}_0(\mathbb{C}, \epsilon)$  is well-defined, i.e. independent of the choice of  $(V, \theta)$ .

The goal of this paper is to prove the following theorem.

**Theorem 1.1.** *Let  $F$  be a Galois extension of  $\mathbb{Q}$ , then for  $\tau \in \tilde{L}_0(F(t), \epsilon)$  we get*

$$\begin{aligned} \tau = 0 \in \tilde{L}_0(F(t), \epsilon) \\ \Leftrightarrow \text{sign}(\rho(\tau)(z)) = 0 \text{ for all } \rho \in G_0(F) \text{ and all transcendental } z \in S^1. \end{aligned}$$

This result was stated by Litherland [L84, p. 358], but there is no proof in the literature.

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## 2. TOPOLOGICAL MOTIVATION

The motivation for studying  $\tilde{L}_0(F(t))$  comes from knot theory. A knot  $K \subset S^{n+2}$  is a smooth submanifold homeomorphic to  $S^n$ . A knot is called slice if it bounds a smooth  $(n+1)$ -disk in  $D^{n+3}$ . For even dimensions Kervaire [K65] showed that all knots are slice and Levine [L69] showed that in the case  $n > 1$  a knot is slice if and only if it is algebraically slice, i.e. if its Seifert form is metabolic.

The case  $n = 1$  turned out to be much harder. A significant breakthrough was made by Casson and Gordon [CG86] who found examples of knots that are algebraically slice but not geometrically slice. They used a sliceness obstruction which lies in  $\tilde{L}_0(F(t))$  where  $F/\mathbb{Q}$  is a number field. We give a quick exposition of their results.

Let  $K \subset S^3$  be an oriented knot. Denote by  $M_K$  the result of zero framed surgery along  $K$ . There exists a canonical isomorphism  $\epsilon : H_1(M_K) \rightarrow \mathbb{Z}$ . Let  $k$  be some prime power, denote by  $M_k$  the  $k$ -fold cover corresponding to  $H_1(M_K) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/k$ . Denote by  $TH_1(M_k)$  the  $\mathbb{Z}$ -torsion part of  $H_1(M_k)$ .

Let  $\chi : H_1(M_k) \rightarrow TH_1(M_k) \rightarrow S^1$  be a character of order  $m$ . Set  $F := \mathbb{Q}(e^{2\pi i/m})$ . Since  $\Omega_3(\mathbb{Z} \times \mathbb{Z}/m) = H_3(\mathbb{Z} \times \mathbb{Z}/m)$  is torsion (cf. [CF64]) there exists  $(V_k^4, \epsilon \times \chi)$  and some  $r \in \mathbb{N}$  such that  $\partial(V_k, \epsilon \times \chi) = r(M_k, \epsilon \times \chi)$ . Let  $\alpha : \mathbb{Z} \times S^1 \rightarrow \text{Aut}(F(t))$  be the map given by  $\alpha(n, z)(f(t)) := t^n z f(t)$ . Casson and Gordon showed that the  $F(t)$ -valued pairing on  $H_2^{\alpha \circ (\epsilon \times \chi)}(V_k, F(t))$  is non-singular and therefore defines an

element  $t(V_k) \in L_0(F(t))$ . Denote by  $t_0(V_k)$  the image of the ordinary intersection pairing on  $H_2(V_k)$  under the canonical map  $L_0(\mathbb{Z}) \rightarrow L_0(F(t))$ .

**Theorem 2.1.** [CG86] *Let  $K$  be a knot,  $k, m$  prime powers and  $\chi : H_1(L_k) \rightarrow S^1$  a character of order  $m$ .*

1.

$$\tau(K, \chi) := (t(V_k) - t_0(V_k)) \otimes \frac{1}{r} \in \tilde{L}_0(F_\chi(t))$$

*is a well-defined invariant of  $(M_k, \epsilon \times \chi)$ , i.e. independent of the choice of  $V_k$ .*

2.  $H_1(M_k) = \mathbb{Z} \oplus TH_1(M_k)$ .

3. *If furthermore  $K$  is slice then there exists a subgroup  $Q \subset TH_1(M_k)$  with  $|Q|^2 = |TH_1(M_k)|$  such that for any  $\chi : TH_1(M_k) \rightarrow S^1$  of order  $m$  vanishing on  $Q$  we get  $\tau(K, \chi) = 0$ .*

In [F03] the author shows that theorem 2.1 can be completely reformulated in terms of eta invariants. This proof relies in particular on the fact that elements in  $\tilde{L}_0(F(t))$  can be detected by signature invariants, i.e. the proof relies on theorem 1.1.

### 3. PROOF OF THE MAIN THEOREM

3.1. **The groups  $\tilde{L}_0(F), \tilde{L}_0(F(t))$ .** We quote a result from Ranicki [R98, p. 493].

**Proposition 3.1.** *The following map is an isomorphism*

$$\begin{aligned} \tilde{L}_0(F) &\rightarrow \bigoplus_{\rho \in G_0(F)} \tilde{L}_0(\bar{\mathbb{Q}}) \\ \tau \otimes r &\mapsto (\rho(\tau) \otimes r)_{\rho \in G_0(F)} \end{aligned}$$

Consider the case where  $F := \mathbb{Q}[t]/q(t)$ ,  $q(t)$  irreducible and  $q(t) = uq(t^{-1})$  for some unit  $u \in \mathbb{Q}[t^{-1}, t]$ . Then there's an involution given by  $\bar{t} = t^{-1}$ , which is non-trivial if  $q(t) \neq t - 1, t + 1$ . In this case the set  $G(F)$  corresponds canonically to the set of all roots of  $q(t)$  lying in  $S^1$  and  $G_0(F)$  corresponds to all roots  $z \in S^1$  of  $q(t)$  with  $\text{Im}(z) \geq 0$ .

The goal of this section is to prove the following theorem.

**Theorem 3.2.** *Let  $F$  be a Galois extension of  $\mathbb{Q}$ , then for  $\tau \in \tilde{L}_0(F(t), \epsilon)$  we get*

$$\tau = 0 \in \tilde{L}_0(F(t), \epsilon) \Leftrightarrow \rho(\tau) = 0 \in \tilde{L}_0(\bar{\mathbb{Q}}(t), \epsilon) \text{ for all } \rho \in G_0(F)$$

To simplify the notation we'll only prove the case  $\epsilon = 1$ . We need some definitions and results from [R98, ch. 39C].

*Definition.* Let  $F$  be a field with a possibly trivial involution. Then define  $LAut_{fib}^0(F, \epsilon)$  to be the Witt group of triples  $(V, \theta, f)$  where  $V$  is a vector space over  $F$ ,  $\theta$  an  $\epsilon$ -hermitian form on  $V$  and  $f$  an isometry of  $(V, \theta)$  such that  $(f - 1)$  is an automorphism as well. Let  $\tilde{L}Aut_{fib}^0(F, \epsilon) := LAut_{fib}^0(F, \epsilon) \otimes \mathbb{Q}$ .

**Proposition 3.3.** [R98, p. 533] *Let  $F$  be a field with (possibly trivial) involution.*

1. *There exists a split exact sequence*

$$0 \rightarrow \tilde{L}_0(F[t, t^{-1}], \epsilon) \rightarrow \tilde{L}_0(F(t), \epsilon) \rightarrow \tilde{L}Aut_{fib}^0(F, -\epsilon) \rightarrow 0$$

2. *Denote by  $\overline{\mathfrak{M}}(F)$  the set of irreducible monic polynomials  $p(t)$  in  $F[t]$  with the added property that  $\overline{p(t)} = up(t)$  for some unit  $u \in F[t, t^{-1}]$  and  $\overline{\mathfrak{M}}^0(F) := \overline{\mathfrak{M}}(F) \setminus \{t-1\}$ . For  $p(t) \in \overline{\mathfrak{M}}^0(F)$  define*

$$\begin{aligned} r_{p(t)} : \tilde{L}Aut_{fib}^0(F, \epsilon) &\rightarrow \tilde{L}_0(F[t, t^{-1}]/p(t), \epsilon) \\ (V, \theta, f) \otimes r &\rightarrow (Ker\{p(f) : V \rightarrow V\}, \tilde{\theta}) \otimes r \end{aligned}$$

where  $\tilde{\theta}(a, b) = \sum_{i=0}^{deg(p)-1} \theta(a, bt^i)t^{-i}$  and  $t$  acts by  $f$ . Then

$$\prod_{p(t) \in \overline{\mathfrak{M}}^0(F)} r_{p(t)} : \tilde{L}Aut_{fib}^0(F, \epsilon) \xrightarrow{\cong} \bigoplus_{p(t) \in \overline{\mathfrak{M}}^0(F)} \tilde{L}_0(F[t, t^{-1}]/p(t), \epsilon)$$

is an isomorphism and the inverse map is given by

$$\begin{aligned} \tilde{L}_0(F[t, t^{-1}]/p(t), \epsilon) &\rightarrow \tilde{L}Aut_{fib}^0(F, \epsilon) \\ (V, \theta) \otimes r &\mapsto (V, tr_{(F[t, t^{-1}]/p(t))/F} \circ \theta, t) \otimes r \end{aligned}$$

3. *The map*

$$\begin{aligned} \tilde{L}_0(F, \epsilon) &\rightarrow \tilde{L}_0(F[t, t^{-1}], \epsilon) \\ (V, \theta) \otimes r &\mapsto ((V, \theta) \otimes_F F[t, t^{-1}]) \otimes r \end{aligned}$$

is an isomorphism.

There exists a commuting diagram of exact sequences ( $G_0 = G_0(F)$ )

$$\begin{array}{ccccccc} 0 & \rightarrow & \tilde{L}_0(F[t, t^{-1}]) & \rightarrow & \tilde{L}_0(F(t)) & \rightarrow & \tilde{L}Aut_{fib}^0(F, -1) \rightarrow 0 \\ & & \downarrow \prod_{\rho \in G_0} \rho & & \downarrow \prod_{\rho \in G_0} \rho & & \downarrow \prod_{\rho \in G_0} \rho \\ 0 & \rightarrow & \bigoplus_{\rho \in G_0} \tilde{L}_0(\overline{\mathbb{Q}}[t, t^{-1}]) & \rightarrow & \bigoplus_{\rho \in G_0} \tilde{L}_0(\overline{\mathbb{Q}}(t)) & \rightarrow & \bigoplus_{\rho \in G_0} \tilde{L}Aut_{fib}^0(\overline{\mathbb{Q}}, -1) \rightarrow 0 \end{array}$$

From propositions 3.1 and 3.3 it follows that the first vertical map is an injection. Once we show that the last vertical map is an injection as well it follows that the middle vertical map is an injection, this will prove theorem 3.2.

For  $p \in F[t, t^{-1}]$  irreducible we'll write  $F_p := F[t, t^{-1}]/p(t)$ . Note that there exists a canonical correspondence

$$\begin{aligned} \{(\rho, z) | \rho \in G(F) \text{ and } z \in S^1 \text{ such that } \rho(p)(z) = 0\} &\leftrightarrow G(F_p) \\ (\rho, z) &\mapsto (\rho_z : \sum a_i t^i \rightarrow \rho(a_i) z^i) \end{aligned}$$

since  $F/\mathbb{Q}$  is Galois. Consider

$$\begin{aligned} \tilde{L}Aut_{fib}^0(F, \epsilon) &\xrightarrow{\cong} \bigoplus_{p \in \overline{\mathfrak{M}}^0(F)} \tilde{L}_0(F_p, \epsilon) \hookrightarrow \bigoplus_{\rho \in G_0} \bigoplus_{p \in \overline{\mathfrak{M}}_0(F)} \bigoplus_{\substack{z \in S^1 \setminus \{1\} \\ \rho(p)(z) = 0}} \tilde{L}_0(\overline{\mathbb{Q}}, \epsilon) \xrightarrow{\mu_{\rho,z}} \mathbb{Q} \\ &\downarrow \prod_{\rho \in G_0} \rho \\ \bigoplus_{\rho \in G_0} \tilde{L}Aut_{fib}^0(\overline{\mathbb{Q}}, \epsilon) &\xrightarrow{\cong} \bigoplus_{\rho \in G_0} \bigoplus_{z \in S^1 \setminus \{1\}} \tilde{L}_0(\overline{\mathbb{Q}}, \epsilon) \xrightarrow{\sigma_{\rho,z}} \mathbb{Q} \end{aligned}$$

where  $\mu_{\rho,z}$  and  $\sigma_{\rho,z}$  denotes the composition of projection maps on the corresponding  $\tilde{L}_0(\overline{\mathbb{Q}}, \epsilon)$  summand and taking signatures. Note that  $\mu_{\rho,z}$  is well-defined, since different  $p(t)$ 's have disjoint zero sets. Define  $\mu_{\rho,z}$  to be the zero map if  $z$  is not a root for any  $\rho(p(t))$ .

**Proposition 3.4.** *Let  $p \in \overline{\mathfrak{M}}_0(F)$ . For  $(V, \theta) \in \tilde{L}_0(F_p, \epsilon)$  we get  $\sigma_{\rho,z}(V, \theta) = \mu_{\rho,z}(V, \theta)$  for all  $\rho \in G_0(F)$  and  $z \in S^1 \setminus \{1\}$  such that  $\rho(p)(z) = 0$ .*

**Corollary 3.5.** *The map*

$$\prod_{\rho \in G_0(F)} \rho : \tilde{L}Aut_{fib}^0(F, \epsilon) \rightarrow \bigoplus_{\rho \in G_0(F)} \tilde{L}Aut_{fib}^0(\overline{\mathbb{Q}}, \epsilon)$$

is an injection.

Note that this corollary concludes the proof of theorem 3.2. We'll first prove the corollary.

*Proof.* The induced map

$$\prod_{\rho, z \in S^1 \setminus \{1\}} \mu_{\rho,z} : \tilde{L}_0(F_p, \epsilon) \rightarrow \bigoplus_{\rho, z \in S^1 \setminus \{1\}} \mathbb{Q}$$

is an injection. From the proposition it also follows that the induced map

$$\prod_{\rho \in G_0(F), z \in S^1 \setminus \{1\}} \sigma_{\rho,z} : \tilde{L}_0(F_p, \epsilon) \rightarrow \bigoplus_{\rho \in G_0(F), z \in S^1} \mathbb{Q}$$

is an injection. Since different  $p$ 's have disjoint sets of zeros it follows that

$$\prod_{\rho \in G_0(F), z \in S^1 \setminus \{1\}} \sigma_{\rho,z} : \bigoplus_{p \in \overline{\mathfrak{M}}^0(F)} \tilde{L}_0(F_p, \epsilon) \rightarrow \bigoplus_{\rho \in G_0(F), z \in S^1} \mathbb{Q}$$

is an injection as well. But this implies that the intermediate map

$$\prod_{\rho \in G_0(F)} \rho : \tilde{L}Aut_{fib}^0(F, \epsilon) \rightarrow \bigoplus_{\rho \in G_0(F)} \tilde{L}Aut_{fib}^0(\overline{\mathbb{Q}}, \epsilon)$$

is injective. □

Now we'll prove proposition 3.4.

*Proof.* Let  $p \in \overline{\mathfrak{M}}_0(F)$ . Denote the zeros of  $\rho(p)(t)$  by  $\alpha_1, \dots, \alpha_n$ . Pick a zero  $\alpha$  of  $\rho(p)$ , we can assume that  $\alpha = \alpha_1$ . Denote the induced embedding  $F_p \rightarrow \overline{\mathbb{Q}}$  by  $\rho_\alpha$ . Consider

$$\begin{array}{ccccccc} \tilde{L}_0(\overline{\mathbb{Q}}, \epsilon) & \xleftarrow{r_{t-z}} & \tilde{L}Aut_{fib}^0(\overline{\mathbb{Q}}, \epsilon) & \xleftarrow{\rho} & \tilde{L}Aut_{fib}^0(F, \epsilon) & \xleftarrow{\tilde{L}_0(F_p, \epsilon)} & \tilde{L}_0(\overline{\mathbb{Q}}, \epsilon) \\ \theta_l & \mapsto & \overline{\mathbb{Q}} \otimes_F (V, tr_{F_p/F} \circ \theta, t) & \mapsto & (V, tr_{F_p/F} \circ \theta, t) & \mapsto & (V, \theta) & \mapsto & \theta_r \end{array}$$

We have to show that  $\text{sign}(\theta_l) = \text{sign}(\theta_r)$ . Note that  $\theta_r$  denotes the form

$$\begin{array}{ccc} \theta_r : V \otimes_{F_p} \overline{\mathbb{Q}} \times V \otimes_{F_p} \overline{\mathbb{Q}} & \rightarrow & \overline{\mathbb{Q}} \\ (v_1 \otimes_{F_p} z_1, v_2 \otimes_{F_p} z_2) & \mapsto & \bar{z}_1 \rho_\alpha(\theta(v_1, v_2)) z_2 \end{array}$$

where  $F_p$  acts on  $\overline{\mathbb{Q}}$  via  $\rho_\alpha$ .

Now we have to understand  $\theta_l$ . In the following we'll view  $\overline{\mathbb{Q}}$  as an  $F$ -module via  $\rho$ . The form  $\overline{\mathbb{Q}} \otimes_F (V, tr_{F_p/F} \circ \theta, t)$  is given by

$$\begin{array}{ccc} V \otimes_F \overline{\mathbb{Q}} \times V \otimes_F \overline{\mathbb{Q}} & \xrightarrow{\text{tr}_{F_p/F} \circ \theta} & \overline{\mathbb{Q}} \otimes_F F \otimes_F \overline{\mathbb{Q}} & \rightarrow & \overline{\mathbb{Q}} \\ (v_1 \otimes_F z_1, v_2 \otimes_F z_2) & \mapsto & \bar{z}_1 \otimes_F \text{tr}_{F_p/F}(\theta(v_1, v_2)) \otimes_F z_2 & \mapsto & \bar{z}_1 \rho(\text{tr}_{F_p/F}(\theta(v_1, v_2))) z_2 \end{array}$$

Denote by  $\overline{\mathbb{Q}}_p$  the ring  $F_p \otimes_F \overline{\mathbb{Q}} = \overline{\mathbb{Q}}[t, t^{-1}]/\rho(p(t))$ . It is easy to see that the map

$$\overline{\mathbb{Q}}_{p(t)} = F_p \otimes_F \overline{\mathbb{Q}} \xrightarrow{\text{tr}_{F_p/F} \otimes \text{id}} F \otimes_F \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}$$

coincides with  $\text{tr}_{\overline{\mathbb{Q}}_p/\overline{\mathbb{Q}}} : \overline{\mathbb{Q}}_p \rightarrow \overline{\mathbb{Q}}$ . Therefore the form  $\overline{\mathbb{Q}} \otimes_F (V, tr_{F_p/F} \circ \theta, t)$  is given by

$$\begin{array}{ccc} V \otimes_F \overline{\mathbb{Q}} \times V \otimes_F \overline{\mathbb{Q}} & \rightarrow & \overline{\mathbb{Q}}_p & \xrightarrow{\text{tr}_{\overline{\mathbb{Q}}_p/\overline{\mathbb{Q}}}} & \overline{\mathbb{Q}} \\ (v_1 \otimes_F z_1, v_2 \otimes_F z_2) & \mapsto & \theta(v_1, v_2) \otimes_F \bar{z}_1 z_2 & \mapsto & \text{tr}_{\overline{\mathbb{Q}}_p/\overline{\mathbb{Q}}}(\theta(v_1, v_2)) \otimes_F \bar{z}_1 z_2 \end{array}$$

We can write  $V \otimes_F \overline{\mathbb{Q}} = V_1 \oplus \dots \oplus V_n$  where  $V_i := \text{Ker}\{(t - \alpha_i) : V \otimes_F \overline{\mathbb{Q}} \rightarrow V \otimes_F \overline{\mathbb{Q}}\}$  since the minimal polynomial of  $t$  is  $p(t) = \prod_{i=1}^n (t - \alpha_i)$ . Then  $\theta_l$  is given by restricting the above form to  $V_1$ .

We can decompose the  $\overline{\mathbb{Q}}[t]$ -module  $\overline{\mathbb{Q}}_p = \overline{\mathbb{Q}}[t]/\rho(p(t))$  as follows

$$\overline{\mathbb{Q}}_p = \bigoplus_{i=1}^n \text{Ker}\{(t - \alpha_i) : \overline{\mathbb{Q}}_p \rightarrow \overline{\mathbb{Q}}_p\} = \bigoplus_{i=1}^n \overline{\mathbb{Q}}_i$$

where  $\overline{\mathbb{Q}}_i := \text{Ker}\{(t - \alpha_i) : \overline{\mathbb{Q}}_p \rightarrow \overline{\mathbb{Q}}_p\}$ . Note that  $\dim_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}_i) = 1$ . Consider the map  $\mu_{\alpha_i} : \overline{\mathbb{Q}}_i \rightarrow \overline{\mathbb{Q}}$  given by  $p(t) \mapsto p(\alpha_i)$ . These maps define isomorphisms of  $\overline{\mathbb{Q}}$ -algebras. Then the trace function is given by

$$\begin{array}{ccc} \text{tr}_{\overline{\mathbb{Q}}_p/\overline{\mathbb{Q}}} : \bigoplus_{i=1}^n \overline{\mathbb{Q}}_i = \overline{\mathbb{Q}}_p & \rightarrow & \overline{\mathbb{Q}} \\ (z_1, \dots, z_n) & \mapsto & \sum_{i=1}^n \mu_i(z_i) \end{array}$$

since  $\text{tr}_{\overline{\mathbb{Q}}_p/\overline{\mathbb{Q}}} = \text{tr}_{(\bigoplus_{i=1}^n \overline{\mathbb{Q}}_i)/\overline{\mathbb{Q}}} = \sum_{i=1}^n \text{tr}_{\overline{\mathbb{Q}}_i/\overline{\mathbb{Q}}}$ . The form  $\theta \otimes_F \overline{\mathbb{Q}} : V \otimes_F \overline{\mathbb{Q}} \times V \otimes_F \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$  restricts to a form  $V_1 \times V_1 \rightarrow \overline{\mathbb{Q}}_1$  and  $\theta_l$  is given by  $V_1 \times V_1 \rightarrow \overline{\mathbb{Q}}_1 \xrightarrow{\text{tr}} \overline{\mathbb{Q}}$ .

We can now compute  $\theta_l$ . Let  $\sum_{j=1}^{s_1} v_{1j} \otimes_F z_{1j}, \sum_{l=1}^{s_2} v_{1l} \otimes_F z_{1l} \in V_1$ , then

$$\begin{aligned} \theta_l(\sum_{j=1}^{s_1} v_{1j} \otimes_F z_{1j}, \sum_{l=1}^{s_2} v_{1l} \otimes_F z_{1l}) &= \text{tr}_{\bar{\mathbb{Q}}_p/\bar{\mathbb{Q}}}(\sum_{j=1}^{s_1} \sum_{l=1}^{s_2} \theta(v_{1j}, v_{2l}) \otimes_F \bar{z}_{1j} z_{2l}) = \\ &= \mu_\alpha(\sum_{j=1}^{s_1} \sum_{l=1}^{s_2} \rho(\theta(v_{1j}, v_{2l})) \bar{z}_{1j} z_{2l}) = \\ &= \sum_{j=1}^{s_1} \sum_{l=1}^{s_2} \rho_\alpha(\theta(v_{1j}, v_{2l})) \bar{z}_{1j} z_{2l} \end{aligned}$$

Consider the following sequence of canonical isomorphisms:

$$V_1 = \text{Ker}\{(t - \alpha) : V \otimes_F \bar{\mathbb{Q}} \rightarrow V \otimes_F \bar{\mathbb{Q}}\} \cong (V \otimes_F \bar{\mathbb{Q}}) \otimes_{\bar{\mathbb{Q}}[t]} \bar{\mathbb{Q}} \cong V \otimes_{F[t]} \bar{\mathbb{Q}} \cong V \otimes_{F_p} \bar{\mathbb{Q}}$$

The resulting isomorphism is given by

$$\begin{aligned} V_1 = \text{Ker}\{(t - \alpha) : V \otimes_F \bar{\mathbb{Q}} \rightarrow V \otimes_F \bar{\mathbb{Q}}\} &\cong V \otimes_{F_p} \bar{\mathbb{Q}} \\ \sum_{j=1}^s v_j \otimes_F z_j &\mapsto \sum_{j=1}^s v_j \otimes_{F_p} z_j \end{aligned}$$

It now follows immediately that the forms  $\theta_l, \theta_r$  are isomorphic.  $\square$

**3.2. The group  $\tilde{L}_0(\bar{\mathbb{Q}}(t))$ .** We need to quote one more fact.

**Proposition 3.6.** [R98, p. 533] *Let  $F$  be a field. The splitting  $\tilde{L}Aut_{fib}^0(F, -\epsilon) \rightarrow \tilde{L}_0(F(t), \epsilon)$  in the exact sequence*

$$0 \rightarrow \tilde{L}_0(F[t, t^{-1}], \epsilon) \rightarrow \tilde{L}_0(F(t), \epsilon) \rightarrow \tilde{L}Aut_{fib}^0(F, -\epsilon) \rightarrow 0$$

is given by

$$(V, \theta, f) \mapsto (V \otimes_F F(t), (v, w) \rightarrow (1 - t^{-1})\theta((1 - f)^{-1}v, w) + \epsilon(1 - t)\overline{\theta((1 - f)^{-1}w, v)})$$

**Theorem 3.7.** *Let  $\tau \in \tilde{L}_0(\bar{\mathbb{Q}}(t))$ , then*

$$\tau = 0 \in \tilde{L}_0(\bar{\mathbb{Q}}(t)) \Leftrightarrow \tau(z) = 0 \in \tilde{L}_0(\bar{\mathbb{Q}}) \text{ for all transcendental } z.$$

Combining theorems 3.2 and 3.7 we now get a proof for theorem 1.1.

*Proof.* Proposition 3.3, part (1) and (3), shows that there exists an isomorphism

$$\tilde{L}_0(\bar{\mathbb{Q}}) \oplus \tilde{L}Aut_{fib}^0(\bar{\mathbb{Q}}, -1) \rightarrow \tilde{L}_0(\bar{\mathbb{Q}}(t))$$

Let  $Z := S^1 \setminus \{1\} \cap \bar{\mathbb{Q}}$ , then  $\bar{\mathfrak{M}}_0(\bar{\mathbb{Q}}) = \{t - z | z \in Z\}$ . Using that  $\tilde{L}_0(\bar{\mathbb{Q}}) \cong \mathbb{Q}$  via the signature and using part (2) of proposition 3.3 we get isomorphisms

$$\begin{aligned} \bigoplus_{z \in Z} \mathbb{Q} &\xrightarrow{\cong} \bigoplus_{z \in Z} \tilde{L}_0(\bar{\mathbb{Q}}[t, t^{-1}]/(t - z), -1) \xrightarrow{\cong} \tilde{L}Aut_{fib}^0(\bar{\mathbb{Q}}, -1) \\ (r_z)_{z \in Z} &\mapsto \bigoplus (\bar{\mathbb{Q}}[t, t^{-1}]/(t - z), i) \otimes r_z \mapsto \bigoplus (\bar{\mathbb{Q}}, i, z) \otimes r_z \end{aligned}$$

The isomorphisms above and proposition 3.6 show that in  $\tilde{L}_0(\bar{\mathbb{Q}}(t))$  the form  $\tau$  is equivalent to

$$(\bar{\mathbb{Q}}(t), 1) \otimes r_0 \oplus \bigoplus_{j=1}^s (\bar{\mathbb{Q}}(t), i(1 - t^{-1})(1 - \bar{z}_j)^{-1} + i(1 - t)(1 - z_j)^{-1}) \otimes r_j$$

where  $z_j \in S^1 \setminus \{1\}, j = 1, \dots, s$  are distinct and  $r_0, \dots, r_s \in \mathbb{Q}$ . Note that  $\tau = 0$  if and only if  $r_0 = r_1 = \dots = r_s = 0$ .

We can assume that  $r_i \in \mathbb{N}$  for all  $i$ , and hence restrict ourselves to forms in  $L_0(\mathbb{Q}(t))$ . Then the matrix

$$A(t) := (1) \otimes r_0 \oplus \bigoplus_{j=1}^s (i(1-t^{-1})(1-\bar{z}_j)^{-1} + i(1-t)(1-z_j)^{-1}) \otimes r_j$$

represents  $\tau$ . The signature function  $z \mapsto \text{sign}(A(z))$  is locally constant, its only jumps are when  $\det(A(t)) = 0$ , i.e. when

$$(1-t^{-1})(1-\bar{z}_j)^{-1} + (1-t)(1-z_j)^{-1} = 0 \text{ for some } j$$

i.e. when  $t = \frac{1-\bar{z}_j}{1-z_j} \in S^1$ . It is clear that  $\text{sign}(A(1)) = r_0$  and that the jump of the signature function at  $t = \frac{1-\bar{z}_j}{1-z_j} \in S^1$  is  $2r_j$ . The theorem follows now easily since the transcendental numbers in  $S^1$  are dense.  $\square$

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