

Summary.

In my research I mostly apply algebraic methods to study problems in low-dimensional topology. More explicitly, I try to address the following problems:

- (1) Find criteria for whether a given knot $K \subset S^3$ bounds an embedded disk in D^4 .
- (2) Find methods for computing the Thurston norm of a given 3-manifold.
- (3) Find methods for determining whether a given 3-manifold is fibered or not.
- (4) Study Taubes' conjecture in symplectic geometry using algebraic methods.
- (5) Relate the Cheeger–Gromov L^2 -eta invariant to Blanchfield forms.
- (6) Find obstructions for the existence of smooth 4-manifolds with intersection forms over a group ring.

In the following I will first outline some of the work I have done over the last couple of years. I will then list various problems I have been thinking about or which I hope to attack in the near future.

Recent Research: Slice knots.

A knot $K \subset S^3$ is called slice, if it bounds a smooth 2-disk in D^4 . In higher odd dimensions Levine [Le69], [Le69b] found a computable algebraic method of determining whether a knot is slice or not. In 1975 Casson and Gordon [CG86] first found examples which show that the high dimensional results (which relied on the Whitney trick) can not be extended to the case of one dimensional knots. Recently Cochran, Orr and Teichner [COT03] constructed an infinite series of obstructions to finding slice disks for a given knot and used L^2 -eta invariants to detect highly non-trivial examples.

In my thesis [Fr04] I used (finite dimensional) eta invariants, introduced by Atiyah, Patodi and Singer [APS75], to give obstructions to a knot being slice. I also show that these obstructions contain and slightly extend the obstructions of Casson and Gordon (cf. also [Fr05e]). It follows from work of Taehee Kim [Ki04] and myself [Fr04] that the eta invariant obstruction is independent of the L^2 -eta invariant obstruction, i.e. there are knots which have non-zero eta invariants but zero L^2 -eta invariants and vice versa.

An important subclass of slice knots is given by ribbon knots. In [Fr04] I showed that eta invariants also give obstructions to a knot being ribbon. It was expected that the (metabelian) L^2 -eta invariant ribbon-obstructions are stronger than the more classical eta invariant ribbon-obstructions, but I showed in [Fr05] that in fact the eta invariant obstructions subsume the L^2 -eta invariant obstructions.

In further work [Fr05b] I used eta invariants to construct link concordance invariants which are stronger than any previously known invariants. Using an explicit formula

from [Fr05b] and an algorithm for generating interesting boundary links [Fr03] I found in [Fr05b] examples of links, which have the same Alexander polynomial and the same one-dimensional eta invariants, but which are not link concordant.

A knot is called topologically slice if it bounds a locally flat disk in D^4 . All the sliceness obstructions mentioned above are in fact obstructions to a knot being topologically slice. Together with Peter Teichner I proved in [FT05] a partial converse to the above obstructions: We found criteria which are sufficient to show that certain knots are in fact topologically slice. More precisely, we used Freedman's surgery theory in dimension 4 and non-commutative algebra to show that a knot bounds a locally flat disk D with $\pi_1(D^4 \setminus D) \cong \mathbb{Z}$ or $\pi_1(D^4 \setminus D) \cong \mathbb{Z} \times_{\gamma} \mathbb{Z}[\frac{1}{2}]$ if and only if a certain Blanchfield form vanishes.

This result is the first extension of Freedman's result from 1983 that a knot with trivial Alexander polynomial is topologically slice. We also state an elegant conjecture which might eventually provide a complete criterion for determining whether a given knot is topologically slice or not.

In a future paper [CFT06] Tim Cochran, Peter Teichner and I will show that a satellite construction gives under certain circumstances new examples of topologically slice knots and links which are not smoothly slice.

Recent Research: Thurston norm and fibered 3-manifolds.

Let M be a compact 3-manifold. Thurston [Th86] defined in 1976 a norm on $H^1(M; \mathbb{R})$ which measures the complexity of surfaces dual to elements in $H^1(M; \mathbb{Z})$. For example if M is a knot complement, then the Thurston norm is determined by the knot genus.

It is a classical result that the degree of the Alexander polynomial gives a lower bound on the knot genus. This was extended by McMullen [Mc02] who used the multivariable Alexander polynomial to define the Alexander norm on $H^1(M; \mathbb{R})$ which gives a lower bound on the Thurston norm.

Given a 3-manifold M and $\phi \in H^1(M; \mathbb{Z})$ Cochran [Co04] and Harvey [Ha05][Ha06] considered the degrees of 'higher order one-variable Alexander polynomials' defined over non-commutative Laurent polynomial rings to get a sequence of invariants $\delta_n(M, \phi)$ which they showed to be a never decreasing sequence of numbers which give lower bounds on the Thurston norm. Furthermore these invariants equal the Thurston norm if M fibers over ϕ .

In my work on the Thurston norm and fibered manifolds I use Reidemeister torsion. Reidemeister torsion can be viewed as a 'better behaved' close relative of the (higher order) Alexander polynomials. In [Fr05c] I reformulated and extended the invariants of Cochran and Harvey in terms of Reidemeister torsion. In [FK05c] Taehee Kim and I used Reidemeister torsion and a certain Cohn localization to show that any jump in the sequence $\delta_n(M, \phi)$ is necessarily even, answering in particular a question of Cochran [Co04].

There is no non-commutative analogue of the multivariable Alexander polynomial, but in [FH06] Shelly Harvey and I still manage to define a higher order analogue of McMullen’s Alexander norm using non-commutative rings. This allows us to completely determine the Thurston norm in many examples. As a corollary to this result we show that Harvey’s maps $\phi \mapsto \delta_n(M, \phi)$ (cf. [Ha05]) are norms for each n . This had been an open problem for several years.

McMullen’s results can also be extended using twisted Reidemeister torsion over commutative fields. The invariants can be computed efficiently, for example using the program ‘KnotTwister’ which I wrote [Fr05d]. Together with Taehee Kim I define in [FK05] and [FK05b] a twisted version of the Alexander norm and we show that it determines the Thurston norm in many cases, in particular using ‘KnotTwister’ we show that it determines the genus for all knots with up to 12 crossings. In [FK05] we also show that twisted Reidemeister torsion can be used to give very strong fibering obstructions, for example we detect all non-fibered knots with up to 12 crossings.

Recent Research: Symplectic manifolds and Taubes’ conjecture.

Let M be a closed 3-manifold. Taubes conjectured that if $S^1 \times M$ is symplectic, then (M, ϕ) fibers over S^1 for some $\phi \in H^1(M; \mathbb{Z})$.

Using results of Taubes and Meng–Taubes it was shown by Kronheimer [Kr98] and Vidussi [Vi03] that if $S^1 \times M$ is symplectic, then there exists a $\phi \in H^1(M; \mathbb{Z})$ such that the corresponding Alexander polynomial is monic and determines the Thurston norm of ϕ . This supports Taubes’ conjecture since the same conclusion holds if (M, ϕ) fibers over S^1 .

In [FK05] we showed that if M is fibered then certain twisted Alexander polynomials are monic. Using a finite-cover trick we show in [FV06], in joint work with Stefano Vidussi, that a large number of these twisted Alexander polynomials are also monic if $S^1 \times M$ is symplectic.

In [FV06b] (cf. also [FV06c]) we prove Taubes’ conjecture in the case that M has vanishing Thurston norm, for example if M is the zero-framed surgery along a genus one knot. This answers in particular a question of Peter Kronheimer. Furthermore in [FV06b] we reduce Taubes’ conjecture to Thurston’s conjecture that the fundamental groups of hyperbolic manifolds are subgroup separable.

Future research plans.

The following is a list of projects which I intend to pursue in the coming months and years.

- (1) In joint work with Stefano Vidussi we hope to prove Taubes’ conjecture in all cases. We are following up on several ideas which already allows us to go beyond the significant progress we have made so far in our papers.

- (2) The results of [FT05] suggest the possibility that a certain Blanchfield form contains all the information on whether a knot is topologically slice or not. I intend to study the relationship between this Blanchfield pairing and the obstruction theory in [COT03].
- (3) I hope to show that the L^2 -eta invariant for $(M^3, \pi_1(M) \rightarrow G)$, where G is a torsion-free amenable group, is determined by a certain Blanchfield form. This question has been asked by Teichner [Te02]. Note that this is known in the case $G = \mathbb{Z}$. The idea is to use ideas of [BDH80] to find a supgroup H of G which is again torsion-free amenable and a pair $(W, \pi_1(W) \rightarrow H)$ such that $\partial W = M$ and $\pi_1(M) \rightarrow G \rightarrow H$ extends over $\pi_1(W)$. The intersection pairing on W then determines the L^2 -eta invariant, it then remains to show that it determines the Blanchfield form (this is known to be true for $H = \mathbb{Z}$).
- (4) Jake Rasmussen introduced a completely combinatorial invariant based on Khovanov homology which gives obstructions to a knot being smoothly slice. I intend to study the results by Jake Rasmussen to determine whether it is possible to construct an invariant which can distinguish between smoothly slice knots and ribbon knots. It has been an open question for over 30 years whether the class of slice knots and the class of ribbon knots are the same. One of the main obstacles so far has been the lack of useful invariants.
- (5) Many of my results (e.g. [FH06], [Fr05c], [FK05c]) relied heavily on the fact that I was using Reidemeister torsion instead of Alexander polynomials. The reason why Reidemeister torsion ‘behaves better’ is that it considers ‘all of M ’ and not just its first homology. I think that a similar logic applied to pairings (replacing the Blanchfield pairing of a 3-manifold by Ranicki’s symmetric signature) will prove equally fruitful.
- (6) The indeterminacy of higher order Alexander polynomials is too large to make sense of ‘monicness’ which is a prerequisite for a strong fiberedness obstruction theorem. I hope to address this problem by using Reidemeister torsion over skew fields which has a much smaller indeterminacy. I then hope to show that Reidemeister torsion over skew fields gives an interesting fiberedness obstruction.
- (7) The method of using non-commutative rings as in [COT03], [Co04], [Ha05] can be applied to many other problems. For example it is possible to get lower bounds on the number of critical points of an \mathbb{R} -valued respectively S^1 -valued Morse function.
- (8) It is known that most Bing doubles of knots are not slice, but all the known invariants are inconclusive for the Bing double of the Figure 8 knot. I intend to use twisted Reidemeister torsion (over non-commutative rings if necessary) to address this question.

- (9) Together with Ian Hambleton, Paul Melvin, Peter Teichner and I am trying to find obstructions for the realization of intersection forms by smooth 4-manifolds over a group ring. The idea is to apply Donaldson's results to finite covers of the 4-manifold.
- (10) Stefano Vidussi and I [FV06a] showed that twisted Alexander polynomials detect the unknot and that in many cases twisted Alexander polynomials detect whether a knot is fibered or not. In future work we hope to show that the twisted Alexander polynomials corresponding to $SL(2, \mathbb{C})$ -representations also detect the genus of a knot.

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