RESEARCH STATEMENT

FEI XU

1. Introduction

Recall that a variety is said to be *rational* if it is birational to $\mathbb{P}^n$. For many years, mathematicians have worked on the rationality of smooth complete intersections, but currently very few of them are known to be rational. In his thesis, M. Reid [Rei72] showed that the nonsingular intersection of two quadrics $Q_1 \cap Q_2$ is always rational. Explicitly, in an odd dimensional projective space $\mathbb{P}^n$, the intermediate Jacobian of the general complete intersection is isomorphic (as a principally polarized abelian variety) to the ordinary Jacobian of the hyperelliptic curve which is the double cover of $\mathbb{P}^1$ branched over the $n+1$ points corresponding to the singular quadrics in the pencil. The birational mapping $\mathbb{P}^{n-2} \dashrightarrow X$ blows up this curve.

A *Grassmannian* is a space which parametrizes all linear subspaces of a vector space of a given dimension. The Grassmannian $Gr(2, n)$ is the space of 2-planes in an n dimensional space. The *Plücker embedding* realizes $Gr(2, n)$ as a subvariety of the projective space $\mathbb{P}(\wedge^2 k^n)$ of the second exterior power of the n dimensional space.

Let X be a smooth linear section of the Grassmannian $Gr(2, n)$ of codimension r in its Plücker embedding in projective space. Inspired by Reid's results, I analyze the rationality of X and its Hodge structure in the cases of even and odd codimensions. The Hodge diamond of X sheds light on what subvarieties are blown up in the birational parametrization.

Throughout, I assume the ground field k to be the field of complex numbers $\mathbb{C}$.

2. Current Work

2.1. The rationality of the smooth linear sections X of Gr(2,n).

As a guiding example, we look at linear sections X of $Gr(2, 6)$ for various values of the codimension r. Note that the dimension of $Gr(2, 6)$ is 8. In this case, the rationality of the linear section X is presented below:

- when $r \leq 4$, X is a rational variety;
- when $r = 5$, Takeuchi [Tak89] and Iskovskih [Isk79] constructed two different birational maps from X to a cubic 3-fold. Clemens and Griffiths [CG72] showed that X is not rational, by using Hodge theory and intermediate Jacobian;
- when $r = 6$, X is a $K3$ surface with trivial canonical bundle;
- when $r = 7$, X is a curve of genus 8, which has positive canonical bundle.

In general, if $X \subset Gr(2,n)$ is a codimension r smooth linear section, adjunction formula tells us that $K_X = \mathcal{O}(-n+r)$. We thus have the following :

- if $n < r < \frac{n^2-n-2}{2}$, X is of general type;
- if $r = n$, X is Calabi-Yau;
- if $r \leq n-1$, X is Fano.

Moreover, we also have the following rationality theorem:

Theorem 2.1. *In the case described above, if $r \leq n-2$, X is rational.*

Let's look at the following example:

Example 2.2. *When $r = 4, n = 5$, the codimension 4 linear section X in $Gr(2,5)$ is a del Pezzo surface of degree 5. By the classification of del Pezzo surfaces, X is given by blowing up 4 points in $\mathbb{P}^2$ with no 3 points on a line, therefore it is rational.*

2.2. **The pattern of Hodge diamonds.** Note that all Fano manifolds of middle index and Picard number no less than 2 have been completely classified [Wis94]. If $X \subset Gr(2,n)$ is a codimension r linear section, then for $r = 4$ and 5, X are exactly the index $\frac{\dim X}{2}$ and $\frac{\dim X + 1}{2}$ cases respectively with Picard number one.

The stratification of the Grassmannians and Lefschetz Hyperplane section theorem give all the Hodge numbers of the linear section X except the middle row of the Hodge diamond. We have the precise pattern for the following case:

Theorem 2.3. *When $r = 3$ and $n = 2m+1$ is odd, the Hodge diamond of X has no nontrivial middle cohomology classes. The Hodge diamond is displayed in Figure 1.*

We consider codimension 3 smooth linear section X of $Gr(2,n)$ when $n = 2m$ is even. For the Hodge diamond of X in this case:

- all entries off the middle row can be obtained by Lefschetz hyperplane section theorem;
- a similar argument as Theorem 3.4 shows that the middle Betti number $h^{4m-7} = m^2 - 3m + 2$;
- existing examples suggest certain distribution pattern of Hodge numbers in the middle row and the expected Hodge diamond is displayed in Figure 2.

3. Future Work

3.1. **More patterns of Hodge diamonds.** Can we verify the expected pattern of the Hodge diamond in the $r = 3, n = 2m$ case (§2.2) and generalize these results to the codimension $r = 4$ case? In order to do so, I need more techniques to compute the middle row of the Hodge diamond. We address the problem of determining the Hodge polynomials of complete intersections in homogeneous spaces. Specifically, let X be a homogeneous variety under a connected linear algebraic group G, and let $Y_1, ..., Y_r \subset X$ be smooth hypersurfaces in general position; then $Y := Y_1 \cap ... \cap Y_r$ is a smooth complete intersection in X. There are algorithms to compute the Hodge polynomial of Y in the cases that X is a projective space $\mathbb{P}^n$ [Hir66] or a torus $(\mathbb{C}^*)^n$ [DK86]. More recently, M. Brion [Bri09] addressed the case $X = G/H$, where $[P,P] \subset H \subset P$ for some parabolic subgroup P of G. In our situation: X is the Grassmannian $Gr(2,n)$ and $Y = Y_1 \cap ... \cap Y_r$ where Y_i $(1 \leq i \leq r)$ is a specialization of hyperplane section of X. To compute all the Hodge numbers of Y, it suffices to compute its Hodge polynomial

$$e_Y(-1,v) = \sum_q \chi(Y, \Omega_Y^q) v^q,$$

because the only unknown Hodge numbers are in the middle row of the Hodge diamond.

3.2. **Finding factorizations for rational linear sections.** I analyzed the rationality of the codimension r linear sections X of the Grassmannian $Gr(2,n)$ in terms of r and n (§2.1). When X is rational, one would like to get a decomposition of the birational map into "elementary links", i.e., explicit blowups and blowdowns.

Our analysis of Hodge structures constrains the subvarieties that may arise as centers of these blowups. For instance, when cohomology consists only of (p,p) clasees, are these centers necessarily rational?

References

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FIGURE 1. for $r=3, n=2m+1$, the Hodge diamond of X

$$
\begin{array}{c}
1 \\
0 \qquad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \qquad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \qquad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
\iddots \qquad\qquad \vdots \qquad\qquad \ddots \\
0 \qquad\qquad 0 \quad m-1 \quad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \quad 0 \qquad 0 \quad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \quad m-1 \quad 0 \qquad\qquad 0 \\
0 \qquad \cdots \qquad 0 \quad 0 \qquad 0 \quad 0 \qquad \cdots \qquad 0 \\
0 \qquad\qquad 0 \quad m-1 \quad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \quad 0 \qquad 0 \quad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \quad m-1 \quad 0 \qquad\qquad 0 \\
\ddots \qquad\qquad \vdots \qquad\qquad \iddots \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \qquad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \qquad 0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \qquad 0 \\
1
\end{array}
$$

FIGURE 2. for $r = 3, n = 2m$, the expected Hodge diamond of X

$$
\begin{array}{c}
1 \\
0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
\iddots \qquad \qquad \vdots \qquad \qquad \ddots \\
0 \qquad \qquad m-1 \qquad \qquad 0 \\
0 \quad \cdots \quad \frac{(m-1)(m-2)}{2} \qquad \frac{(m-1)(m-2)}{2} \quad \cdots \quad 0 \\
0 \qquad \qquad m-1 \qquad \qquad 0 \\
\ddots \qquad \qquad \vdots \qquad \qquad \iddots \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \\
1
\end{array}
$$