

Math 102 Fall 2008 Exam 1

Instructor: F. Xu

Wednesday, October 1, 2008

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **fifty minutes**. Do all 5 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work or explanation will receive little to no credit.

Please print your name clearly here.

Print name: _____

Upon finishing please sign the pledge below:

On my honor I have neither given nor received any aid on this exam.

Grader's use only:

1. _____ /10

2. _____ /10

3. _____ /10

4. _____ /10

5. _____ /10

1. [10 points]

Compute the integral

$$\int \ln x dx.$$

We integrate by parts using $u = \ln x$ and $v' = 1$ so that $u' = 1/x$ and $v = x$. So we get

$$\int \ln x dx = x \ln x - \int \frac{1}{x} x dx = x \ln x - x + C.$$

2. [10 points]

Compute the integral

$$\int x^3 \sqrt{1-x^2} dx.$$

One could first start by trying to simplify the integrand using integration by parts. But eventually we will have to make a trigonometric substitution so we might as well do it at the beginning.

We let $x = \sin t$ so that $dx = \cos t dt$. Then we get

$$\int x^3 \sqrt{1-x^2} dx = \int \sin^3 t \sqrt{1-\sin^2 t} \cos t dt = \int \sin^3 t \cos^2 t dt.$$

Now we rewrite $\sin^3 t = \sin t \sin^2 t = \sin t(1 - \cos^2 t)$ to get

$$\int \sin t(1 - \cos^2 t) \cos^2 t dt = \int \sin t \cos^2 t dt - \int \sin t \cos^4 t dt.$$

The reason for doing this last step is that now we can substitute $u = \cos t$ and $du = -\sin t dt$ to get

$$\int (-u^2) du - \int (-u^4) du = -\frac{u^3}{3} + \frac{u^4}{4} + C$$

Substituting back we get

$$-\frac{\cos^3 t}{3} + \frac{\cos^4 t}{4} + C = -\frac{(\cos \sin^{-1} x)^3}{3} + \frac{(\cos \sin^{-1} x)^4}{4} + C$$

Now $\cos \sin^{-1} x = \sqrt{1-x^2}$ so we get

$$-\frac{(1-x^2)^{3/2}}{3} + \frac{(1-x^2)^2}{4} + C.$$

3. [10 points]

Compute the integral

$$\int \frac{x^2 + 1}{(x + 1)(x^2 + 2x + 3)} dx.$$

Notice that we cannot factor $x^2 + 2x + 3$ so we try to rewrite the integrand as

$$\frac{A}{x + 1} + \frac{Bx + C}{x^2 + 2x + 3}.$$

So we get

$$\frac{x^2 + 1}{(x + 1)(x^2 + 2x + 3)} = \frac{A(x^2 + 2x + 3) + (Bx + C)(x + 1)}{(x + 1)(x^2 + 2x + 3)}$$

which gives (after expanding $(Bx + C)(x + 1) = Bx^2 + x(B + C) + C$ and collecting terms)

$$x^2 + 1 = x^2(A + B) + x(2A + B + C) + (3A + C)$$

This means that $A + B = 1$, $2A + B + C = 0$ and $3A + C = 1$.

From the first of these we get $B = 1 - A$ and substituting into the second we get $2A + 1 - A + C = 0$ or $A + C = -1$. Together with $3A + C = 1$ this gives $A = 1$ and $C = -2$ so that $B = 0$.

Now $\int \frac{1}{x+1} dx = \ln(1 + x) + C$. On the other hand,

$$\int \frac{-2}{x^2 + 2x + 3} dx = \int \frac{-2}{(x + 1)^2 + 2} dx = \int \frac{-1}{\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1}.$$

where we substituted $u = \frac{x+1}{\sqrt{2}}$.

Thus

$$\int \frac{-2}{x^2 + 2x + 3} dx = -\tan^{-1}\left(\frac{x + 1}{\sqrt{2}}\right) + C$$

and the final answer is

$$\ln(1 + x) - \tan^{-1}\left(\frac{x + 1}{\sqrt{2}}\right) + C.$$

4. [10 points] Determine whether or not the following integral converges. If it converges find its value. If it diverges to $\pm\infty$ indicate which one.

$$\int_0^1 \frac{\cos x}{\sin x} dx.$$

It is important here to realize that $\sin x = 0$ when $x = 0$ in the interval $[0, 1]$.

Now, substituting $u = \sin x$ we get

$$\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln u + C = \ln \sin x + C.$$

Notice that $\sin x > 0$ when $0 < x \leq 1$ so we get

$$\int_t^1 \frac{\cos x}{\sin x} dx = [\ln \sin(x)]_t^1 = \ln \sin(1) - \ln \sin(t)$$

for $t > 0$. Now as $t \rightarrow 0^+$ then $\sin(t) \rightarrow 0^+$ so that $\ln \sin(t) \rightarrow -\infty$ so that

$$\lim_{t \rightarrow 0^+} \ln \sin(1) - \ln \sin(t) \rightarrow \infty$$

This means that the integral diverges to ∞ .

5. [10 points]

Compute the integral

$$\int \frac{1}{\sin \theta} d\theta.$$

Multiply top and bottom by $\sin \theta$ to get

$$\int \frac{\sin \theta}{\sin^2 \theta} d\theta = \int \frac{\sin \theta}{1 - \cos^2 \theta} d\theta.$$

Now we substitute $u = \cos \theta$ and $du = -\sin \theta d\theta$ to get

$$\int \frac{-du}{1 - u^2} = -\int \frac{1}{1 - u^2} du$$

Now $\frac{1}{1-u^2} = \frac{1/2}{1-u} + \frac{1/2}{1+u}$ so that we get

$$-\left(-\frac{1}{2} \ln(1 - u) + \frac{1}{2} \ln(1 + u)\right)$$

and substituting back $u = \cos \theta$ we get

$$\frac{1}{2} \ln(1 - \cos \theta) - \frac{1}{2} \ln(1 + \cos \theta) + C.$$