Math 102 Spring 2008: **Solutions: HW #3**

Instructor: Fei Xu

1. section 7.5, #18 Evaluate

$$\int \frac{x+1}{x^3-x^2}dx.$$

We'll solve using partial fractions. If we assume $\frac{x+1}{x^3-x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$, clearing denominators gives us $Ax^2 - Ax + Bx - B + Cx^2 = x + 1$. Then associating like terms gives us that $A + C = 0$, $-A + B = 1$, and $-B = 1$. Therefore, $B = -1$, $A = -2$, and $C = 2$. Then we have

$$\frac{x+1}{x^3-x^2}dx = \int \left(\frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2} \right) dx = 2\ln|x-1| - 2\ln|x| + \frac{1}{x} + K.$$

2. section 7.5, #34 Evaluate

$$\int \frac{x^2+4}{(x^2+1)^2(x^2+2)}dx.$$

Again, we'll use partial fractions. If we assume that $\frac{x^2+4}{(x^2+1)^2(x^2+2)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$, then clearing denominators gives us that

$$\begin{aligned} A(x^5 &+ 2x^3 + x) + B(x^4 + 2x^2 + 1) + C(x^5 + 3x^3 + 2x) + D(x^4 + 3x^2 + 2) \\ &\quad + E(x^3 + 2x) + F(x^2 + 2) \\ &= (A + C)x^5 + (B + D)x^4 + (2A + 3C + E)x^3(2B + 3D + F)x^2 \\ &\quad + (A + 2C + 2E)x + (B + 2D + 2F) \\ &= x^2 + 4 \end{aligned}$$

Therefore, we have that

$$\begin{aligned} &A + C = 0, && B + D = 0, && 2A + 3C + E = 0 \\ &2B + 3D + F = 1, && A + 2C + 2E = 0, && B + 2D + 2F = 4 \end{aligned}$$

Solving for this system, we get that $A = 0$, $B = 2$, $C = 0$, $D = -2$, $E = 0$, and $F = 3$. Therefore,

$$\begin{aligned} \int \frac{x^2+4}{(x^2+1)^2(x^2+2)}dx &= \int \left(\frac{2}{x^2+2} - \frac{2}{x^2+1} + \frac{3}{(x^2+1)^2} \right) dx \\ &= \sqrt{2}\arctan\left(\frac{\sqrt{2}}{2}x\right) - \frac{1}{2}\arctan x + \frac{3}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + C \end{aligned}$$

For the last integral on the right, we use trigonometric substitution, let $u = \tan\theta$, then

$$\begin{aligned}
\int \frac{3}{(x^2+1)^2} dx &= \int \frac{3\sec^2\theta}{\sec^4\theta}\ d\theta \\
&= 3\int \frac{1}{\sec^2\theta}\ d\theta \\
&= \frac{3}{2}(\theta + \sin\theta\cos\theta) \\
&= \frac{3}{2}\left(\arctan x + \frac{x}{x^2+1}\right)
\end{aligned}$$

3. section 7.5, #38 Make a preliminary substitution before using the method of partial fractions to evaluate

$$\int \frac{\cos\theta}{\sin^2\theta - \sin\theta - 6} dx$$

Let's use the substitution $u = \sin\theta$, so $du = \cos\theta\ d\theta$. Then we have

$$\frac{\cos\theta}{\sin^2\theta - \sin\theta - 6} dx = \int \frac{1}{u^2 - u - 6} du$$

Then using partial fractions, let $\frac{1}{u^2-u-6} = \frac{A}{u-3} + \frac{B}{u+2}$, so we get that $(A+B)u + (2A - 3B) = 1$, and so $A = \frac{1}{5}$, $B = -\frac{1}{5}$. Then we have

$$\begin{aligned}
\frac{\cos\theta}{\sin^2\theta - \sin\theta - 6} dx &= \int \frac{1}{u^2-u-6} du \\
&= \frac{1}{5}\int \left(\frac{1}{u-3} - \frac{1}{u+2}\right) du \\
&= \frac{1}{5}(\ln|u-3| - \ln|u+2|) + C \\
&= \frac{1}{5}(\ln|\sin\theta - 3| - \ln|\sin\theta + 2| + C
\end{aligned}$$

4. section 7.6, #2 Use trigonometric substitution to evaluate

$$\int \frac{1}{\sqrt{4-9x^2}} dx$$

Let $x = \frac{2}{3}\sin u$. Then we have

$$4 - 9x^2 = 4 - 4\sin^2 u = 4(1 - \sin^2 u) = 4\cos^2 u$$

and $dx = \frac{2}{3}\cos u\ du$. This gives us that

$$\begin{aligned}\int \frac{1}{\sqrt{4-9x^2}}dx &= \frac{2}{3}\int \frac{1}{2\cos u}\cos u\ du \\ &= \frac{1}{3}u + C \\ &= \frac{1}{3}\arcsin\frac{3x}{2} + C\end{aligned}$$

5. section 7.6, #12 Use trigonometric substitution to evaluate

$$\int \frac{x^3}{\sqrt{x^2+25}}dx$$

Let $x = 5\tan u$, so we have that

$$x^2 + 25 = 25\tan^2 u + 25 = 25\left(\sec^2 u - 1\right) + 25 = 25\sec^2 u$$

and $du = 5\sec^2 u\ du$. This gives us that

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+25}}dx &= \int \frac{125\tan^3 u}{5\sec u}\cdot 5\sec^2 u\ du \\ &= 125\int \tan^3 u\ \sec u\ du \\ &= 125\int \left(\sec^2 u - 1\right)\tan u\ \sec u\ du\end{aligned}$$

Now make the substitution $w = \sec u$, so $dw = \sec u\ \tan u\ du$. Then we have

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+25}}dx &= 125\int \left(\sec^2 u - 1\right)\tan u\ \sec u\ du \\ &= 125\int \left(w^2 - 1\right)dw \\ &= 125\left(\frac{1}{3}w^3 - w\right) + C \\ &= 125\left(\frac{1}{3}\sec^3 u - \sec u\right) + C\end{aligned}$$

Note that the right triangle with angle u, opposite side x, and adjacent side 5 has hypotenuse $\sqrt{x^2+25}$. Therefore, we have that $\sec u = \frac{\sqrt{x^2+25}}{5}$, so we get

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+25}}dx &= 125\left(\frac{1}{3}\cdot\frac{\left(x^2+25\right)^{3/2}}{125} - \frac{\left(x^2+25\right)^{1/2}}{5}\right) + C \\ &= \left(x^2+25\right)^{1/2}\left(\frac{1}{3}\left(x^2+25\right) - 25\right) \\ &= \frac{1}{3}\left(x^2+25\right)^{1/2}\left(x^2-50\right) + C\end{aligned}$$

6. section 7.6, #16 Use trigonometric substitution to evaluate

$$\int \sqrt{1+4x^2}dx$$

Let $x = \frac{1}{2}\tan u$, so we have $dx = \frac{1}{2}\sec^2 u\; du$ and

$$\sqrt{1+4x^2} = \sqrt{1+\tan^2 u} = \sqrt{1+(\sec^2 u - 1)} = \sec u$$

Then we have

$$\begin{aligned}\int \sqrt{1+4x^2}dx &= \frac{1}{2}\int \sec^3 u\; du \\ &= \frac{1}{4}\sec u\ \tan u + \frac{1}{4}\ln|\sec u + \tan u| + C\end{aligned}$$

from integral formula for secant. Now note that the right triangle with angle u, opposite side $2x$, and adjacent side 1 has hypotenuse $\sqrt{1+4x^2}$, so $\sec u = \sqrt{1+4x^2}$ and $\tan u = 2x$, and so

$$\begin{aligned}\int \sqrt{1+4x^2}dx &= \frac{1}{4}\sec u\ \tan u + \frac{1}{4}\ln|\sec u + \tan u| + C \\ &= \frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln\left|\sqrt{1+4x^2}+2x\right| + C\end{aligned}$$

7. section 7.6, #24 Use trigonometric substitution to evaluate

$$\int \frac{1}{(4-x^2)^2}dx$$

Let $x = 2\sin u$, so $4 - x^2 = 4 - 4\sin^2 u = 4\cos^2 u$ and $dx = 2\cos u\; du$. Then

$$\begin{aligned}\int \frac{1}{(4-x^2)^2}dx &= \int \frac{2\cos u}{16\cos^4 u}du \\ &= \frac{1}{8}\int \sec^3 u\; du \\ &= \frac{1}{16}\sec u\ \tan u + \frac{1}{16}\ln|\sec u + \tan u| + C\end{aligned}$$

Now consider the right triangle with angle u, opposite side x, and hypotenuse 2. This has adjacent side $\sqrt{4-x^2}$, so $\sec u = \frac{2}{\sqrt{4-x^2}}$ and $\tan u = \frac{x}{\sqrt{4-x^2}}$. Then we get that

$$\begin{aligned}\int \frac{1}{(4-x^2)^2}dx &= \frac{1}{16}\cdot\frac{2}{\sqrt{4-x^2}}\cdot\frac{x}{\sqrt{4-x^2}} + \frac{1}{16}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + C \\ &= \frac{x}{8(4-x^2)} + \frac{1}{32}\ln\left|\frac{(2+x)^2}{(2+x)(2-x)}\right| + C \\ &= \frac{x}{8(4-x^2)} + \frac{1}{32}\ln\left|\frac{2+x}{2-x}\right| + C\end{aligned}$$

8. section 7.6, #36 Use trigonometric substitution to evaluate

$$\int (4x^2 - 5)^{3/2} dx$$

Let $x = \frac{\sqrt{5}}{2} \sec u$, so $dx = \frac{\sqrt{5}}{2} \sec u \ \tan u \ du$, and

$$4x^2 - 5 = 5\sec^2 u - 5 = 5\tan^2 u.$$

Therefore,

$$\begin{aligned}
\int (4x^2 - 5)^{3/2} dx &= \int \left(5^{3/2} \tan^3 u\right) \cdot \frac{\sqrt{5}}{2} \sec u \ \tan u \ du \\
&= \frac{25}{2} \int \tan^4 u \ \sec u \ du \\
&= \frac{25}{2} \int \left(\sec^4 u - 2\sec^2 u + 1\right) \sec u \ du \\
&= \frac{25}{2} \int \left(\sec^5 u - 2\sec^3 u + \sec u\right) du
\end{aligned}$$

From secant integral formula, we have

$$\begin{aligned}
\int \sec^5 u \ du &= \frac{1}{4} \sec^3 u \ \tan u + \frac{3}{4} \int \sec^3 u \ du \\
&= \frac{1}{4} \sec^3 u \ \tan u + \frac{3}{8} \sec u \ \tan u + \frac{3}{8} \ln|\sec u + \tan u| + C
\end{aligned}$$

Then

$$\begin{aligned}
\int (4x^2 - 5)^{3/2} dx &= \frac{25}{2} \int \left(\sec^5 u - 2\sec^3 u + \sec u\right) du \\
&= \frac{25}{2} \left(\frac{1}{4} \sec^3 u \ \tan u - \frac{5}{8} \sec u \ \tan u + \frac{3}{8} \ln|\sec u + \tan u|\right) + C
\end{aligned}$$

Now consider the right triangle with angle u, adjacent side $\sqrt{5}$, and hypotenuse $2x$. It has opposite side $\sqrt{4x^2 - 5}$, and so $\sec u = \frac{2x}{\sqrt{5}}$ and $\tan u = \frac{\sqrt{5}}{\sqrt{4x^2-5}}$. This gives us

$$\begin{aligned}
\int (4x^2 - 5)^{3/2} dx &= \frac{25}{2} \left(\frac{1}{4} \cdot \frac{8x^3}{5\sqrt{5}} \cdot \frac{\sqrt{4x^2 - 5}}{\sqrt{5}} - \frac{5}{8} \cdot \frac{2x\sqrt{4x^2 - 5}}{5} + \frac{3}{8} \ln\left|\frac{2x + \sqrt{4x^2 - 5}}{\sqrt{5}}\right|\right) + C \\
&= x^3\sqrt{4x^2 - 5} - \frac{25}{8} x\sqrt{4x^2 - 5} + \frac{75}{16} \ln\left|2x + \sqrt{4x^2 - 5}\right| + C
\end{aligned}$$

9. section 7.6, #44 Compute the arc length of the parabola $y = x^2$ over the interval $[0, 1]$.

With $y = x^2$, we have

$$ds = \sqrt{1 + (dy/dx)^2}dx = \sqrt{1 + 4x^2}dx$$

so we want to calculate $L = \int_0^1 \sqrt{1 + 4x^2}\ dx$. Therefore, let $x = \frac{1}{2}\tan\theta$, so then $1 + 4x^2 = 1 + \tan^2\theta = \sec^2\theta$, and $dx = \frac{1}{2}\sec^2\theta$. Now we have

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + 4x^2}\ dx \\ &= \int_0^1 \sec\theta \cdot \frac{1}{2}\sec^2\theta d\theta \\ &= \frac{1}{2}\int_0^1 \sec^3\theta d\theta \\ &= \frac{1}{4}(\sec\theta\ \tan\theta + \ln|\sec\theta + \tan\theta|)\Big|_0^1 \\ &= \frac{1}{4}\left[2x\sqrt{1 + 4x^2} + \ln\left(2x + \sqrt{1 + 4x^2}\right)\right]_0^1 \\ &= \frac{1}{4}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] \end{aligned}$$

10. section 7.7, #6

The derivative of $\sqrt{3 - 2x - x^2}$ is $\frac{-1-x}{\sqrt{3-2x-x^2}}$. So

$$\int \frac{x + 3}{\sqrt{3 - 2x - x^2}}dx = -\sqrt{3 - 2x - x^2} + \int \frac{2}{\sqrt{3 - 2x - x^2}}dx$$

Now we complete the square to get $3 - 2x - x^2 = 4 - (x+1)^2$. Substituting $u = x + 1$ we get

$$\int \frac{2}{\sqrt{3 - 2x - x^2}}dx = \int \frac{2du}{\sqrt{4 - u^2}} = \int \frac{2du}{2\sqrt{1 - (u/2)^2}}$$

Substituting $v = u/2$ we get

$$\int \frac{2dv}{\sqrt{1 - v^2}} = 2\sin^{-1}(v) + C$$

where to get this last equality we either used the substitution $v = \sin\theta$ or remembered the integral from memory.

Substituting back we get

$$2\sin^{-1}(u/2) + C = 2\sin^{-1}(\frac{x + 1}{2}) + C$$

so the final answer is

$$-\sqrt{3 - 2x - x^2} + 2\sin^{-1}(\frac{x + 1}{2}) + C.$$

11. section 7.7, #24

One can do this integral by integration by parts. But let's do it by substituting $x = \sin\theta$. We get

$$\int \frac{\sin^3\theta}{(\cos^2\theta)^4}\cos\theta d\theta = \int \frac{\sin^3\theta}{\cos^7\theta}d\theta.$$

Now $\sin^3\theta = \sin\theta(1-\cos^2\theta)$ so we get

$$\int \frac{\sin\theta}{\cos^7\theta} - \frac{\sin\theta}{\cos^5\theta}d\theta$$

Now we substitute $u = \cos\theta$ to get

$$\int \frac{-du}{u^7} - \frac{-du}{u^5} = \frac{1}{6u^6} - \frac{1}{4u^4} + C$$

Substituting back we get

$$\frac{1}{6\cos^6\theta} - \frac{1}{4\cos^4\theta} + C = \frac{1}{6(\cos\sin^{-1}x)^6} - \frac{1}{4(\cos\sin^{-1}x)^4}$$

Now $\cos\sin^{-1}(x) = \sqrt{1-x^2}$ so we get

$$\frac{1}{6(1-x^2)^3} - \frac{1}{4(1-x^2)^2}.$$

12. section 7.7, #36

$du = a\cos\theta d\theta$ so the left side equals

$$\begin{aligned}\int \frac{1}{(a^2 - a^2\sin^2\theta)^n}a\cos\theta d\theta &= \int \frac{1}{a^{2n}\cos^{2n}\theta}a\cos\theta d\theta \\ &= \frac{1}{a^{2n-1}}\int \frac{1}{\cos^{2n-1}\theta}d\theta \\ &= \frac{1}{a^{2n-1}}\int \sec^{2n-1}\theta d\theta\end{aligned}$$

13. section 7.8, #20

The function $\frac{x}{x^2-1}$ is undefined at $x = 1$ in the interval $[0, 2]$. Hence we must calculate $\int_0^1 \frac{x}{x^2-1}dx$ and $\int_1^2 \frac{x}{x^2-1}dx$. Now if $x > 1$ then $x^2 - 1 > 0$ so

$$\int \frac{x}{x^2-1}dx = \int \frac{du/2}{u} = 1/2\ln(u) + C = 1/2\ln(x^2-1) + C$$

where we substituted $u = x^2 - 1$. If $0 \le x < 1$ then $x^2 - 1 < 0$ so

$$\int \frac{x}{x^2-1}dx = 1/2\ln(1-x^2) + C$$

.

Now as $b \to 1^-$ we have $1 - x^2 \to 0^+$ and hence $\ln(1-x^2) \to -\infty$. Similarly, as $a \to 1^+$ we have $x^2 - 1 \to 0^+$ and hence $\ln(x^2-1) \to -\infty$. Hence

$$\begin{aligned}\int_0^1 \frac{x}{x^2-1}dx + \int_1^2 \frac{x}{x^2-1}dx &= \lim_{b\to 1^-}[1/2\ln(1-x^2)]0^b + \lim_{a\to 1^*}[1/2\ln(x^2-1)]_a^2 \\ &= (-\infty) - 0 + (1/2\ln(3) - (-\infty)\end{aligned}$$

So the integral does not converge.

14. section 7.8 #32

$\cos x \to 0$ as $x \to \pi/2$ so the problem is at $x = \pi/2$. Now letting $u = \cos x$ we get

$$\int \frac{\sin x}{(\cos x)^{4/3}}dx = \int \frac{-du}{u^{4/3}} = 3u^{-1/3} + C = 3(\cos x)^{-1/3} + C.$$

Now as $x \to \pi/2^-$ we have $\cos(x) \to 0^+$ and $(\cos x)^{-1/3} \to \infty$. Thus the integral does not converge.

15. section 7.8 #52

$1 + x^2 > 0$ so we only have to figure out what happens as $x \to \pm\infty$. Now

$$\int \frac{1+x}{1+x^2}dx = \tan^{-1}(x) + \frac{1}{2}\ln(1+x^2) + C$$

which does not converge since $\ln(1+x^2) \to \infty$ as $x \to \infty$.

On the other hand

$$\begin{aligned}\int_{-t}^t \frac{1+x}{1+x^2}dx &= [\tan^{-1}(x) + \frac{1}{2}\ln(1+x^2)]_{-t}^t \\ &= \tan^{-1}(t) - \tan^{-1}(-t) + \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln(1+(-t)^2) \\ &= \tan^{-1}(t) - \tan^{-1}(-t)\end{aligned}$$

Now as $t \to \infty$ we have $\tan^{-1}(t) \to \pi/2$ and $\tan^{-1}(-t) \to -\pi/2$ which means

$$\lim_{t\to\infty}\int_{-t}^t \frac{1+x}{1+x^2}dx = \pi/2 - (-\pi/2) = \pi.$$