Math 102 Spring 2008: **Solutions: HW #5**

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1. section 10.3, #2

   This is a geometric series with $a = 1$ and $r = \frac{1}{e}$. Since $|r| < 1$ this series converges and the sum is

   $$\sum_{n=0}^{\infty} e^{-n} = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$$

2. section 10.3, #6

   This is a geometric series with $a = 1$ and $r = \frac{-1}{4}$. Since $|r| < 1$ this series converges and the sum is

   $$\sum_{n=0}^{\infty} (\frac{-1}{4})^n = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}$$

3. section 10.3, #14

   First,

   $$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n} = \sum_{n=0}^{\infty} \frac{3^n}{4^n} - \sum_{n=0}^{\infty} \frac{2^n}{4^n}$$

   These are two geometric series, the first with $a = 1$ and $r = \frac{3}{4}$, and the second with $a = 1$ and $r = \frac{3}{4}$. In both cases $|r| < 1$ this series converges and the sum is

   $$\frac{1}{1-\frac{3}{4}} - \frac{1}{1-\frac{1}{2}} = 4 - 2 = 2$$

4. section 10.3, #20

   This series diverges by the nth term test, since

   $$\lim_{n\to\infty} \frac{1}{1+(\frac{9}{10})^n} = \frac{1}{1+0} = 1 \neq 0$$

5. section 10.3, #22

   First,

   $$\sum_{n=1}^{\infty} (\frac{\pi}{e})^n = \sum_{n=0}^{\infty} (\frac{\pi}{e})^n - 1$$

   Now, $sum_{n=0}^{\infty}(\frac{\pi}{e})^n$ is a geometric series with $a = 1$ and $r = \frac{\pi}{e}$, since $\pi > e$, $|r| > 1$ so this series diverges. Thus the sum $\sum_{n=1}^{\infty}(\frac{\pi}{e})^n$ diverges.

6. section 10.3, #28

   By the nth term test this series diverges, since

   $$\lim_{n\to\infty} 2^{\frac{1}{n}} = 1$$

   .

7. section 10.3, #42

$$0.337733773377\cdots = \frac{3377}{10^4}+\frac{3377}{10^8}+\frac{3377}{10^{12}}+\cdots = \frac{\frac{3377}{10000}}{1-\frac{1}{10000}} = \frac{3377}{9999} = \frac{307}{909}.$$

8. section 10.3, #46

The given geometric series has $a = 1-x$ and $r = 1-x$. The series converges when $|r| < 1$, ie $|1-x| < 1$ so $0 < x < 2$. Then for such an x the sum is,

$$\sum_{n=1}^{\infty}(x-1)^n = \frac{x-1}{1-(x-1)} = \frac{x-1}{2-x}$$

.

9. section 10.3, #50

The method of partial fractions gives,

$$\frac{1}{4n^2-1} = \frac{1}{2}\Big(\frac{1}{2n-1}+\frac{-1}{2n+1}\Big)$$

. So the kth partial sum of the series is

$$S_k = \sum_{n=1}^{k}\frac{1}{2}\Big(\frac{1}{2n-1}+\frac{-1}{2n+1}\Big) = \frac{1}{2}\Big(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{2k+1}\Big) = \frac{1}{2}\Big(1-\frac{1}{2k+1}\Big)$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \lim_{k\to\infty} S_k = \frac{1}{2}.$$

10. section 10.3, #58

The method of partial fractions gives,

$$\frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$$

.

So the kth partial sum of the series is

$$
\begin{aligned}
S_k = \sum_{n=1}^{k} \frac{2}{n(n+1)(n+2)} \quad &= \quad \frac{1}{1} - \frac{2}{2} + \frac{1}{3} \\
&+ \quad \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \\
&+ \quad \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \\
&+ \quad \frac{1}{4} - \frac{2}{5} + \frac{1}{6} \\
&+ \quad \frac{1}{5} - \frac{2}{6} + \frac{1}{7} \\
&+ \quad \frac{1}{6} - \frac{2}{7} + \frac{1}{8} \\
&+ \quad \frac{1}{7} - \frac{2}{8} + \frac{1}{9} \\
&+ \quad \frac{1}{8} - \frac{2}{9} + \frac{1}{10} \\
&+ \quad \cdots \\
&+ \quad \frac{1}{k-2} - \frac{2}{k-1} + \frac{1}{k} \\
&+ \quad \frac{1}{k-1} - \frac{2}{k} + \frac{1}{k+1} \\
&+ \quad \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}.
\end{aligned}
$$

Inspect the diagonals running fro the top right to lower left. Those fractions with denominator 3 cancel one another, as do those with denominators $4, 5, 6, \ldots, k-1, k$. So

$$S_k = \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{k+1} - \frac{2}{k+1} + \frac{1}{k+2} = \frac{1}{2} - \frac{1}{k+1} + \frac{1}{k+2}.$$

Thus the sum

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)(n+2)} = \lim_{k\to\infty} S_k = \frac{1}{2}.$$