Math 102 Spring 2008: **Solutions: HW #6**

Instructor: F.Xu

1. section 10.4, #2

   Given $f(x) = \sin x$ and $n = 4$, we have

$$\begin{aligned} f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x \end{aligned}$$

   Therefore $P_4(x) = x - \frac{x^3}{3!}$ and $R_4(x) = \frac{x^5}{5!} \cos z$
   for some number z between 0 and x.

2. section 10.4, #4

   Given $f(x) = (1-x)^{-1}$ and $n = 4$, we have

$$\begin{aligned} f'(x) &= (1-x)^{-2} & f'(0) &= 1 \\ f''(x) &= 2(1-x)^{-3} & f''(0) &= 2 \\ f^{(3)}(x) &= 6(1-x)^{-4} & f^{(3)}(0) &= 6 \\ f^{(4)}(x) &= 24(1-x)^{-5} & f^{(4)}(0) &= 24 \\ f^{(5)}(x) &= 120(1-x)^{-6} \end{aligned}$$

   Therefore $P_4(x) = 1 + x + x^2 + x^3 + x^4$ and $R_4(x) = \frac{x^5}{(1-z)^6}$
   for some number z between 0 and x.

3. section 10.4, #24

   Since

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

   then substitute x with x^3 in the series and get

$$\exp(x) = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \frac{x^{15}}{5!} + ... = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

4. section 10.4, #26

   Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

then substiute x with $x/2$ and yeild

$$\sin\frac{x}{2} = \frac{x}{2} - \frac{x^3}{3!\cdot 8} + \frac{x^5}{5!\cdot 32} - \frac{x^7}{7!\cdot 128} + ... = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!\cdot 2^{2n+1}}$$

5. section 10.4, #32

   Given $f(x) = \sin x$ and $a = \pi/2$, we have

$$\begin{aligned} f(x) &= \sin x & f(a) &= 1 \\ f'(x) &= \cos x & f'(a) &= 0 \\ f''(x) &= -\sin x & f''(a) &= -1 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(a) &= 0 \\ f^{(4)}(x) &= \sin x & f^{(4)}(a) &= 1 \\ f^{(5)}(x) &= \cos x & f^{(5)}(a) &= 0 \\ f^{(5)}(x) &= -\sin x & f^{(6)}(a) &= -1 \end{aligned}$$

   we conclude that $f^{(n)}(a) = 0$ if n is odd, whereas $f^{(n)}(a) = (-1)^{n/2}$ if n is even. So the Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x-\pi/2)^{2n} = 1-\frac{1}{2}(x-\pi/2)^2+\frac{1}{24}(x-\pi/2)^4-\frac{1}{720}(x-\pi/2)^6+...$$

6. section 10.4, #36

   Given $f(x) = 1/(1-x)^2$ and $a = 0$, we have

$$\begin{aligned} f(x) &= (1-x)^{-2} & f(a) &= 1 \\ f'(x) &= 2(1-x)^{-3} & f'(a) &= 2 \\ f''(x) &= 6(1-x)^{-4} & f''(a) &= 6 \\ f^{(3)}(x) &= 24(1-x)^{-5} & f^{(3)}(a) &= 24 \\ f^{(4)}(x) &= 120(1-x)^{-6} & f^{(4)}(a) &= 120 \\ f^{(5)}(x) &= 720(1-x)^{-7} & f^{(5)}(a) &= 720 \\ f^{(6)}(x) &= 5040(1-x)^{-8} & f^{(6)}(a) &= 5040 \end{aligned}$$

   It is clear that $f^{(n)}(x) = (n+1)!$ for $n \geq 0$. Therefore the Taylor series for $f(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + ...$$

7. section 10.4, #42

$$\begin{aligned} D_x \cos x &= D_x(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + ...) \\ &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + ... \\ &= -\sin x \end{aligned}$$

and

$$\begin{aligned} D_x \sin x &= D_x(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + ...) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - ... \\ &= \cos x \end{aligned}$$

8. section 10.4, #52

Given $\alpha = \tan^{-1}(1/5)$

$$\begin{aligned} \tan 2\alpha &= \frac{1/5 + 1/5}{1 - 1/25} = 5/12 \\ \tan 4\alpha &= \frac{5/12 + 5/12}{1 - 25/144} = 120/119 \\ \tan(\pi/4 - 4\alpha) &= \frac{1 - 120/119}{1 + 120/119} = -1/239 \end{aligned}$$

Since

$$\tan(\pi/4 - 4\alpha) = -1/239$$

Then

$$\pi/4 - 4\alpha = -\arctan 1/239$$

Thus

$$4 \arctan 1/5 - \arctan 1/239 = \pi/4.$$

9. section 10.4, #55

Suppose that x is a real number. Choose the integer k so large that $k > |2x|$. Let $L = \frac{|x|^k}{k!}$. Suppose that $n = k + 1$. Then

$$\frac{|x|^n}{n!} = \frac{|x|^{k+1}}{(k+1)!} = \frac{|x|^k}{k!} \cdot \frac{|x|}{k+1} < \frac{L}{2} = \frac{L}{2^{n-k}}$$

because $|2x| < k < k + 1$ and $n - k = 1$. Next assume that

$$\frac{|x|^m}{m!} < \frac{L}{2^{m-k}}$$

for some integer $m > k$. Then

$$\frac{|x|^{m+1}}{(m+1)!} = \frac{|x|^m}{m!} \cdot \frac{|x|}{m+1} < \frac{L}{2^{m-k}} \cdot \frac{1}{2} = \frac{L}{2^{m+1-k}}$$

because $|2x| < k < m$. Therefore, by induction,

$$\frac{|x|^n}{n!} < \frac{L}{2^{n-k}}$$

for every integer $n > k$. Now let $n \to +\infty$ to conclude that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$