Math 102 Spring 2008: **Solutions: HW #7**

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1. section 10.4, #56

   The integral of $\frac{1}{1+t}$ is $\ln(1+x)$.

   Integrating the right side we get

   $$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^n}{n+1} + R_n$$

   where

   $$R_n = \int_0^x \frac{(-1)^{n+1} t^{n+1}}{1+t} dt \leq \int_0^x |(-1)^{n+1} t^{n+1}| dt = \int_0^x t^{n+1} dt = \frac{x^{n+2}}{n+2}$$

   where, to get the first inequality, we used that $0 < x \leq 1$ so that $1+t \geq 1$. Now

   $$\lim_{n \to \infty} \frac{x^{n+2}}{n+2} = 0$$

   if $0 < x \leq 1$ so that $R_n \to 0$. Thus we conclude that

   $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!}$$

   if $0 < x \leq 1$.

2. section 10.4, #58

   The argument in question 56 also works if $-1 \leq x \leq 0$. Substituting $-x$ for x we then get

   $$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$$

   Then

   $$\ln(\frac{1+x}{1-x}) = \ln(1+x) - \ln(1-x) = 2(x + \frac{x^3}{+} \frac{x^5}{5} + \ldots)$$

   where we just added the series for $\ln(1+x)$ and $\ln(1-x)$ term by term.

3. section 10.4, #59

   If we substitute $x = 1$ in problem 56 the error is $\frac{1}{n+2}$ if we use n terms to approximate $\ln(2)$. If we substitute $x = 1/3$ in problem 58 then we also get $\ln(2)$ but the error is now

   $$2\frac{(1/3)^{n+2}}{n+2} = 2\frac{1}{3^{n+2}(n+2)}$$

   (the factor of two accounts for the fact we have two error terms, one from the series for $\ln(1+x)$ and the other from the series for $\ln(1-x)$).

   Clearly, the second estimate is much better if we use the same number of terms (since the error term is much smaller).

4. section 10.5, #2
$\frac{1}{(x+1)^{4/3}}$ is a decreasing function for $x \geq 1$. We also have

$$\int \frac{1}{(x+1)^{4/3}} dx = \int (x+1)^{-4/3} dx = -3(x+1)^{-1/3}$$

and this converges as $x \to \infty$. So by the integral test the given series also converges.

5. section 10.5, #6
$\frac{1}{x(x+1)}$ is a decreasing function for $x \geq 1$. We also have

$$\int \frac{1}{x(x+1)} dx = \int \frac{1}{x} - \frac{1}{x+1} dx = \ln(x) - \ln(x+1) = \ln \frac{x}{x+1} = \ln(1 + \frac{1}{x})$$

and this converges as $x \to \infty$. So by the integral test the given series also converges.

6. section 10.5, #8
$\frac{\ln x}{x}$ is a decreasing function for x large (just take its derivative). We also have

$$\int \frac{\ln x}{x} dx = \frac{1}{2} \ln^2 x$$

and this diverges as $x \to \infty$. So by the integral test the given series also diverges.

7. section 10.5, #18 $\ln \frac{x+1}{x} = \ln(1 + \frac{1}{x})$ is a decreasing function for $x \geq 1$ since $1/x$ is decreasing. We also have

$$\begin{aligned} \int \ln \frac{x+1}{x} dx &= \int_0^x \ln(x+1) dx - \int_0^x \ln x dx \\ &= ((x+1)\ln(x+1) - (x+1)) - (x \ln x - x) \\ &= x(\ln(x+1) - \ln(x)) + \ln(x+1) - 1 \\ &\geq \ln(x+1) - 1 \end{aligned}$$

which diverges as $x \to \infty$. So by the integral test the given series also diverges.

8. section 10.5, #20
$\ln \frac{2^{1/x}}{x^2}$ is a decreasing function for $x \geq 1$. We also have

$$\int \frac{2^{1/x}}{x^2} dx = \int -2^u du = -\frac{2^u}{\ln 2} = -\frac{2^{1/x}}{\ln 2}$$

where we used the substitution $u = 1/x$. Since $1/x \to 0$ as $x \to \infty$ this integral converges and so by the integral test the given series also converges.

9. section 10.5, #24
$\frac{1}{x \ln^3 x}$ is a decreasing function for $x \geq 1$ since $x \ln^3 x$ is increasing. We also have
$$\int \frac{1}{x \ln^3 x} dx = \int \frac{1}{u^3} du = -\frac{1}{2u^2} = -\frac{1}{2 \ln^2 x}$$
where we used the substitution $u = \ln x$. Now $\ln x \to \infty$ so this integral converges as $x \to \infty$. Thus by the integral test the given series converges.

10. section 10.5, #32

The function we should use is $f(x) = e^{-x} \sin x$. Now e^{-x} is decreasing to zero as x goes to infinity. But $\sin(x)$ alternates sign between 1 and -1 so that $f(x)$ is not decreasing (it alternates sign). This means that we cannot apply the integral test (which requires that $f(x)$ be eventually decreasing).