Math 212 Spring 2005 **Exam #2**

Instructions: You have **two hours** to complete this exam. You should work alone, without access to the textbook or class notes. You may not use a calculator. Do not discuss this exam with anyone except your instructor.
This exam consists of 7 questions. You must show your work to receive full credit. Be sure to **indicate your final answer clearly** for each question. Pledge your exam when finished, and include your name and section number on the front of the exam. The exam is due by **Friday, 5 p.m.** Good luck!

1. Find and classify the critical points of $f(x, y) = 3xy - x^2y - xy^2$.

2. Find the extreme values of
$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$
on the region described by $x^2 + y^2 \le 16$.

3. Compute the average of the function $f(x, y) = xe^{y^5}$ over the region
$$D = \{(x, y) |\, 0 \le x \le 3; \sqrt{x/3} \le y \le 1\}.$$

4. Find the volume enclosed by the sphere $x^2 + y^2 + z^2 = 1$ and the cylinder $x^2 + y^2 = \frac{1}{2}$.

5. Let S be the solid body bounded by $x^2 + y^2 = 2, z = -\sqrt{x}, z = \sqrt{y}$ with $x \ge 0, y \ge 0$. Compute
$$\int\int\int_S z \; dx\, dy\, dz.$$

6. Let S be the solid body to the right of the yz–plane which is bounded by the planes $y = x$, $y = -x$ and by $x^2 + y^2 + z^2 = 1$, and $x^2 + y^2 + z^2 = 9$. Compute
$$\int\int\int_S \frac{z^2}{x^2 + y^2 + z^2} \, dx\, dy\, dz.$$

7. Let $\mathbf{c}(t) = \left(2t, \frac{4}{3}t^{\frac{3}{2}}, \frac{1}{2}t^2\right)$. Compute the arc length of the path $\mathbf{c}$ between the points $(0, 0, 0)$ and $(2, \frac{4}{3}, \frac{1}{2})$.

Problem 1 (12 points)

- Set $\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$
- Solve $3y - 2xy - y^2 = 0$; $y(3-2x-y) = 0$

  $3x - x^2 - 2xy = 0$; $x(3-x-2y) = 0$

First equation gives two cases:

- $y = 0 \Rightarrow 3x - x^2 = 0 \Rightarrow x = 3$ or $x = 0$.
- $3-2x-y=0 \Rightarrow y = 3-2x \Rightarrow x(3-x-6+4x) = 0$

  $\Rightarrow x = 0\ (y=3)$ or $x = 1\ (y=1)$

Thus, the critical points are : $(3,0)$; $(0,0)$; $(0,3)$; $(1,1)$.

Compute 2^{nd} order derivatives: $\frac{\partial^2 f}{\partial x^2} = -2y$; $\frac{\partial^2 f}{\partial y^2} = -2x$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 3 - 2x - 2y$$

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4xy - (3-2x-2y)^2$$

Classification:

	f_{xx}	f_{yy}	f_{xy}	D	Type
$(0,0)$	0	0	3	-9	saddle
$(3,0)$	0	-6	-3	-9	saddle
$(0,3)$	-6	0	-3	-9	saddle
$(1,1)$	-2	-2	-1	3	local max

3 points per critical point

Problem 2 (12 points)

Find critical points inside the disk $x^2+y^2 \le 16$.

(3) $\frac{\partial f}{\partial x} = 4x - 4 = 0$; $\frac{\partial f}{\partial y} = 6y = 0 \Rightarrow (1,0)$.

Find critical points on the boundary (circle $x^2+y^2=16$) by using Lagrange multipliers:

solve $\nabla f = \lambda \nabla g$ where $g(x,y) = x^2+y^2$.

$(4x-4, 6y) = \lambda (2x, 2y)$

(4) $\begin{cases} 4x - 4 = 2\lambda x \\ 6y = 2\lambda y \end{cases}$

The second equation gives two cases:

(3)
- $y = 0$, and then $x = \pm 4$ (b/c $x^2+y^2=16$)
- $\lambda = 3 \Rightarrow x = -2$ and $y = \pm\sqrt{12} = \pm 2\sqrt{3}$.

Candidate points: $(1,0)$; $(4,0)$, $(-4,0)$, $(-2, 2\sqrt{3})$, $(-2, -2\sqrt{3})$

Evaluate f at these points:

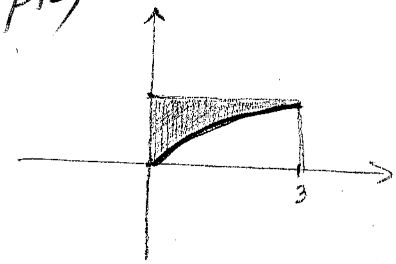
$f(1,0) = -7 \rightarrow$ absolute minimum

(2) $f(4,0) = 11$

$f(-4,0) = 43$

$\left.\begin{array}{l} f(-2, 2\sqrt{3}) = 47 \\ f(-2, -2\sqrt{3}) = 47 \end{array}\right\} \Rightarrow$ absolute maximum.

Problem 3 (16 pts)



$$\text{Average}(f) = \frac{\iint_D f(x,y)\,dx\,dy}{\text{area}(D)} = \frac{\iint_D x\cdot e^{y^5}\,dx\,dy}{\iint_D 1\,dx\,dy}$$

(6) D is given as a y-simple region, but we need to change the order of integration:

$$D: \quad 0 \le y \le 1$$
$$0 \le x \le 3y^2$$

Hence $\iint_D x e^{y^5}\,dx\,dy = \int_0^1 \left(\int_0^{3y^2} x e^{y^5}\,dx \right) dy$

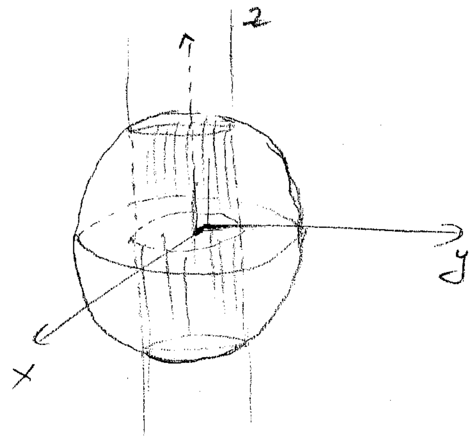
(6) $= \int_0^1 \frac{1}{2} x^2 e^{y^5} \Big|_0^{3y^2} dy = \frac{9}{2} \int_0^1 y^4 e^{y^5}\,dy$

$= \frac{9}{2} \cdot \frac{1}{5} e^{y^5} \Big|_0^1 = \boxed{\frac{9}{10}(e-1)}$

(2) Also $\iint_D 1\,dx\,dy = \int_0^1 \int_0^{3y^2} dx\,dy = \int_0^1 3y^2\,dy = \boxed{1.}$

(2) Hence $\boxed{\text{Average}(f) = \frac{9}{10}(e-1)}$

Problem 4 (16 pts)



In x, y, z coordinates the solid can be described as:

$$x^2+y^2 \le \frac{1}{2}$$

$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$$

It is more convenient to use cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

The limits are: $0 \le r \le \frac{1}{\sqrt{2}}$

(8) $0 \le \theta \le 2\pi$

$$-\sqrt{1-r^2} \le z \le \sqrt{1-r^2}$$

Hence Volume $= \int_{r=0}^{1/\sqrt{2}} \int_{\theta=0}^{2\pi} \int_{z=-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\,dz\,d\theta\,dr$

(8) $= 2\pi \int_0^{1/\sqrt{2}} 2r\sqrt{1-r^2}\,dr$

$$= 2\pi\left[-\frac{2}{3}(1-r^2)^{\frac{3}{2}}\right]_0^{1/\sqrt{2}} = 2\pi\left(-\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} + \frac{2}{3}\right)$$

$$= \frac{4\pi}{3}\left(1 - \frac{1}{2\sqrt{2}}\right).$$

Problem 5 (16 pts)

Limits for S:

$-\sqrt{x} \le z \le \sqrt{y}$

(6) $x^2+y^2 \le 2$ so $0 \le x \le \sqrt{2}$

$x \ge 0,\ y \ge 0$ $\quad 0 \le y \le \sqrt{2-x^2}$

$-\sqrt{x} \le z \le \sqrt{y}$

and $\iiint_S z\,dx\,dy\,dz = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{-\sqrt{x}}^{\sqrt{y}} z\,dz\,dy\,dx$

(10) $= \int_0^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{1}{2}(y-x)\,dy\,dx$

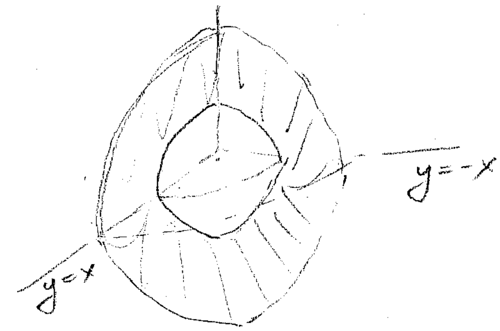
$= \frac{1}{2}\int_0^{\sqrt{2}} \left(\frac{y^2}{2} - xy\right)\Big|_0^{\sqrt{2-x^2}} dx = \frac{1}{2}\int_0^{\sqrt{2}} \frac{(2-x^2)}{2} - x\sqrt{2-x^2}\,dx$

$= \frac{1}{2}\left(x - \frac{x^3}{6} + \frac{(2-x^2)^{\frac{3}{2}}}{3}\right)\Big|_0^{\sqrt{2}} =$

$= \frac{1}{2}\left(\sqrt{2} - \frac{2\sqrt{2}}{6} + 0 - \frac{2\sqrt{2}}{3}\right) = \boxed{0}$

Problem 6 (16 pts)

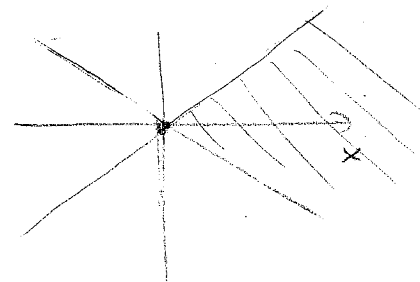
We use spherical coordinates



$$1 \le \rho \le 3$$

$$0 \le \varphi \le \pi \qquad (8)$$

$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$



Hence $\displaystyle\iiint_S \frac{z^2}{x^2+y^2+z^2}\,dx\,dy\,dz$

(8)

$$= \int_{\rho=1}^{3}\int_{\varphi=0}^{\pi}\int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\rho^2\cos^2\varphi}{\rho^2}\,\rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho$$

$$= \frac{\pi}{2}\int_1^3\int_0^{\pi} \rho^2\cos^2\varphi\sin\varphi\,d\varphi\,d\rho = \frac{\pi}{2}\int_1^3 \left(-\rho^2\cdot\frac{\cos^3\varphi}{3}\right)_{\varphi=0}^{\pi} d\rho$$

$$= \frac{\pi}{6}\int_1^3 2\rho^2\,d\rho = \frac{\pi}{3}\,\frac{\rho^3}{3}\Big|_1^3 = \frac{\pi}{3}\left(9-\frac{1}{3}\right)$$

$$= \frac{26\pi}{9}$$

Problem 7 (12 pts)

$$\text{Length}(\bar{c}) = \int_0^1 \|\bar{c}'(t)\| \, dt$$

(the limits $t=0$, $t=1$ are obtained from the

(4) fact that $(0,0,0) \to t=0$

$(2, \frac{4}{3}, \frac{1}{2}) \to t=1$)

(4) $\bar{c}'(t) = (2, 2\sqrt{t}, t)$

$$L(\bar{c}) = \int_0^1 \sqrt{4+4t+t^2} \, dt = \int_0^1 \sqrt{(t+2)^2} \, dt$$

(4)

$$= \int_0^1 (t+2) \, dt = \frac{t^2}{2} + 2t \Big|_0^1 = \underline{\underline{\frac{5}{2}}}$$