

Homework 12, due Wed.Dec.5 :

Exercise 1 Suppose S is a compact connected surface, and $f : S \rightarrow \mathbf{R}$ is a smooth function. For $p \in S$, let $\text{grad } f(p) \in \mathbf{R}^3$ be the unique vector satisfying

$$\langle \text{grad } f(p), v \rangle = df_p(v) \text{ for all } v \in T_p(S) \quad \text{and} \quad \langle \text{grad } f(p), N_p \rangle = 0$$

where N_p is a unit normal to S at p . Suppose F is any extension of f to a smooth function on some open neighborhood of S in \mathbf{R}^3 . Show that

$$\text{grad } f(p) = \nabla F(p) - \langle \nabla F(p), N_p \rangle N_p$$

where $\nabla F(p)$ is the usual gradient in \mathbf{R}^3 , that is, $(\frac{\partial F}{\partial x}(p), \frac{\partial F}{\partial y}(p), \frac{\partial F}{\partial z}(p))$.

In the next 4 exercises, assume $f(x, y, z) = z$ for $(x, y, z) \in S$.

Exercise 2. Draw an example of a compact surface S so that $\text{grad } f$ has exactly 4 zeros, which when listed in vertical order, have indices $+1, -1, -1, +1$.

Exercise 3(corrected). Draw an example of another such S but now so that $\text{grad } f$ has exactly 6 zeros with indices $+1, -1, -1, -1, -1, +1$.

Exercise 4. Draw an example of another such S but now so that $\text{grad } f$ has exactly 6 zeros with indices $+1, +1, -1, -1, +1, +1$.

Exercise 5. Draw an example of another such S but now so that $\text{grad } f$ has exactly 6 zeros with indices $+1, -1, +1, -1, -1, +1$.

Exercise 6. What is the index of the planar vectorfield (x^2, y^3) at $(0, 0)$?

Exercise 7. What is the index of the planar vectorfield $(x^3 - 3xy^2, y^3 - 3x^2y)$ at $(0, 0)$?

For the last 2 exercises it is useful to draw a picture.