Math 423 -Solutions Midterm Exam Fall 2008

1. Suppose that $\mathbf{B}$ is the open unit ball in $\mathbf{R}^3$.

(a) Find the general formula for the solution of

$$\Delta w \;=\; f \text{ on } \mathbf{B}\;,\; w \;=\; g \text{ on } \partial\mathbf{B}\;.$$

Since $\Phi(x) = 1/(4\pi|x|)$, $w(x)$ equals

$$\frac{1}{4\pi}\int_{\mathbf{B}} f(y)\Big[\frac{1}{|y-x|} - \frac{|x|}{\big|\,|x|^2 y - x\big|}\Big]\,dy \;+\; \frac{1-|x|^2}{4\pi}\int_{\partial\mathbf{B}} \frac{g(y)}{|x-y|^3}\,dS_y\;.$$

(b) Find a solution of

$$\Delta\phi \;=\; 0 \text{ on } \mathbf{B}\setminus\{0\} \quad \text{with} \quad \frac{\partial\phi}{\partial\rho} \;=\; 1 \text{ on } \partial\mathbf{B}\;.$$

$-1/|x|$

2. With the same notations as in Problem 1.

(a)) The Divergence Theorem shows that

$$\int_{\partial\mathbf{B}} \frac{\partial v}{\partial\rho}\,dS_y \;=\; \int_{\mathbf{B}} \operatorname{div}(Dv)\,dx \;=\; \int_{\mathbf{B}} \Delta v\,dx \;=\; 0\;.$$

(b) Show that

$$\begin{aligned}
&\int_{\mathbf{B}} u(x,\tau)\,dx \;-\; \int_{\mathbf{B}} u(x,\sigma)\,dx \;=\; \int_{\mathbf{B}}\int_s^t u_t(x,r)\,dr\,dx \\
&= \int_{\mathbf{B}}\int_s^t \Delta u(x,r)\,dr\,dx \;=\; \int_s^t\int_{\mathbf{B}} \operatorname{div}(Du)(x,r)\,dx\,dr \\
&= \int_\sigma^\tau\int_{\partial\mathbf{B}} \frac{\partial u(y,t)}{\partial\rho}\,dS_y\,dt\;.
\end{aligned}$$

3. Suppose that v is harmonic on $\mathbf{R}^2$ and $f = v^2 + \exp(v_{xx}^2)$. Since v_{xx} is harmonic and $t \mapsto t^2$ and $t \mapsto \exp t$ are convex, both v^2 and $\exp(v_{xx}^2)$ are subharmonic. Moreover the sum of 2 subharmonic functions is subharmonic. (Alternately one can show directly that $\Delta f \geq 0$.) A subharmonic function function f satisfies the maximum principle $\max_U |f| = \max_{\partial U}|f|$.

4. Find P.D.E.'s whose general solutions have each of the following forms:

(a) $u(x,t) \;=\; f(x-t)$ for any function f. $u_x + u_t = 0$
(b) $v(x,t) \;=\; f(x-t) \;+\; g(x-2t)$ for any functions $f,\ g$. Here

$$0 \;=\; (\frac{\partial}{\partial x} + \frac{\partial}{\partial t})(2\frac{\partial}{\partial x} + \frac{\partial}{\partial t})v \;=\; 2v_{xx} + 3v_{xt} + v_{tt}\ .$$

(c) $w(x,t) \;=\; f(x-t) + g(x-2t) + x^2$ for any functions $f,\ g$. $2w_{xx}+3w_{xt}+w_{tt} = 4$.

5. Consider a C^2 solution on $\mathbf{R}^2 \times \mathbf{R}$ of the PDE: $\quad v_t = v_{xy}$.

(a) For what constants α, β, γ is the rescaled function $w(x,y,t) = v(\alpha x, \beta y, \gamma t)$ also a solution of this PDE? One computes w_t and w_{xy} using the chain rule to see that the only condition needed is $\gamma = \alpha\beta$. Thus $= v(\alpha x, \beta y, \alpha\beta t)$ is a solution for any α, β.

(b) Find all solutions v of this PDE in the form $X(x)\,Y(y)\,T(t)$ for some C^2 functions X, Y, T of one variable. Since $XYT' = X'Y'T$, we see that T'/T is a constant a and that $T = Ae^{at}$ for some constant A. Also $X'Y' = aXY$ which implies X'/X is a constant b or $X = Be^{bx}$. Also Y'/Y is a constant c or $Y = Ce^{cy}$. Substituting, we obtain the relation $a = bc$. The general solution is $De^{ax}e^{by}e^{abt}$ for any constants D, a, b as is easily checked by substituting.

6. Suppose that a 10 pound turkey is represented by a region U in $\mathbf{R}^3$, and the temperature (in Fahrenheit) $u(x,t)$ at $x \in U$ at time t is given by the standard heat equation. Initially the turkey is at room temperature (75^o) when it is put in a standard oven (not a microwave) that is preheated and controlled to have constant temperature 325^o.

(a) The appropriate PDE initial-boundary value problem describing the cooking of this turkey is $u_t = \Delta u$ on U, $u(x,0) = 75^o$ on U, and $u(x,t) = 325^o$ on ∂U for all $t \geq 0$.

(b) The turkey is cooked when every point of the turkey has temperature at least 175^o. Do we have enough information actually to compute (or numerically approximate) this cooking time? If not, what other information do we need about the turkey? No. One needs to know the Green's function for U which is determined by the *shape* of U.

(c) Suppose we find experimentally that the cooking time of this 10 pound turkey is 4 hours. How long would it take to cook a bigger 20 pound turkey (having the same shape)? If the linear dimensions of the turkey are increased by a factor $\lambda = 2^{1/3}$, then the volume and the weight will be increased by a factor 2 as desired.

Since $u(x/\lambda, t/\lambda^2)$ is a solution using the region λU corresponding to the new turkey, we see that the new cooking time for the same initial and boundary conditions is $2^{2/3} \times (4)$ hours.

7. Suppose $L[u]$ is a linear partial differential operator on a smooth bounded domain U in $\mathbf{R}^n$ and $g(x)$ is a smooth function on the closure of U which vanishes on ∂U. Consider the corresponding evolution problem

$$u_t \;=\; L[u] \text{ on } U \times [0, \infty)\;,\; u(x,t) = g(x) \text{ for either } t = 0 \text{ or } x \in \partial U\;. \qquad (*)\;.$$

(a) Show that if $n = 2$ and $L[u] \equiv u_{xx} + 2u_{yy}$, then the solution of (*) is unique. Notice that $v(x,y,t) = u(x, 2^{-1/2}y, t)$ is a solution of the heat equation with the spatial region $V = \{(x, 2^{1/2}y) : (x,y) \in U\}$ and initial boundary data $h(x,y,t) = g(x, 2^{-1/2}y, t)$. The uniqueness results for v proven in the book determine the same uniqueness for u. Here one can alternately modify the *proof* of the uniqueness for the heat equation to this equation.

(b) In general, assuming (*) is well-posed with a unique smooth bounded solution $u = S(x,t,g)$, find, for any smooth function $f(x,t)$ on $\bar{U} \times [0, \infty)$, a formula (in terms of f and S) for a solution of the inhomogeneous problem

$$v_t - L[v] \;=\; f \text{ on } U\;,\; v(x,t) = 0 \text{ for either } t = 0 \text{ or } x \in \partial U\;.$$

Here we follow Duhamel's Principle and write

$$v(x,t) \;=\; \int_0^t S\big(x, t-s, f(\cdot, s)\big)\, ds\;.$$