Homework 8, due Thursday, Nov.13

Problem 1. Suppose that U is the open interval $(-1,1)$.

(1) Show that the function $v(x) = \operatorname{sgn} x$ is the weak derivative for $u(x) = |x|$.

(2) Show that $v(x) = \operatorname{sgn} x$ itself does not have a weak derivative that is an L^1 function.

Problem 2. Show that the space $C([0,1])$ of continuous functions on $[0,1]$ is a Banach space with the sup norm $\|f\|_{sup} = \sup_{t\in[0,1]} |f(t)|$.

Problem 3. (Now due Nov. 20) Let $\lambda : \mathbf{R}^2 \to [0,+\infty)$ satisfy $\lambda(tx) = |t|\lambda(x)$ for all $t \in \mathbf{R}$ and $x \in \mathbf{R}^2$. Also assume $\lambda(x) \neq 0$ for $x \neq 0$. Show that λ is a *norm* on $\mathbf{R}^2$ if and only if $B = \{x \in \mathbf{R}^2 : \ \lambda(x) < 1\}$ is convex. Hint: A useful fact here is that λ will be a convex function if and only if every set $B_t = \{x \in \mathbf{R}^2 : \ \lambda(x) < t\}$ is convex. This can be checked by restricting to lines.

Problem 4. (Now due Nov. 20) Show that such an open convex B will be the inside of an ellipse if and only if the corresponding norm λ comes from an inner product on $\mathbf{R}^2$. Hint: Starting with the positive definite inner product, one can find orthogonal vectors where the corresponding norm has maximum and minimum values. Use these as a basis to see that B is bounded by an ellipse. Conversely if B is bounded by an ellipse, then one can use the equation for the ellipse to construct an inner product whose unit ball is bounded by the ellipse. Then uniqueness will show that the given norm must come from this inner product.

Problem 5. Suppose $f_j(x) = 1 - j\,|x|$ for $x \in \mathbf{R}^2$ with $|x| \leq 1/j$ and $f_j(x) = 0$ for $|x| \geq 1/j$. Show that $f_j \to 0$ in $L^2(\mathbf{R}^2)$ as $j \to \infty$, but f_j does not approach 0 in $W^{1,2}(\mathbf{R}^2) = H^1(\mathbf{R}^2)$. Also show that $< f_j\,,\, g >_{H^1(\mathbf{R}^2)} \to 0$ for any $g \in C^1_c(\mathbf{R}^2)$. This is called *weak* H^1 *convergence of* f_j *to* 0.