**Existence of a Minimizer with Dirichlet Boundary Data**

Then initial goal in the course is to describe, for a given smooth function $g$ on the boundary of a smoothly bounded domain $U \subset \mathbf{R}^n$, how to obtain in the admissible class

$$\mathcal{A} \;=\; \{u \in \mathcal{C}^\infty(\bar{U}) \,:\, u|\partial U = g\}\ ,$$

a minimizer for a functional $I[u] = \int_U L(Du,u,x)\,dx$ for certain smooth integrands $L : \mathbf{R}^n \times \mathbf{R} \times U \to \mathbf{R}$. This is not true for all integrands, and we will need to impose further conditions (in (2) and (6) below) on $L$. Also this existence problem is usually solved in 3 steps:

Step I. Prove the existence of a minimizer $u$ in a larger admissible class of $W^{1,q}(U)$ Sobolev functions (with $q < n$ depending on $L$).

Step II. Showing that $u$ satisfies (in a weak sense) the Euler-Lagrange Dirichlet problem

$$-\sum_{i=1}^n \big(L_{p_i}(Du,u,x)\big)_{x_i} \;+\; L_z(Du,u,x) \;=\; 0 \text{ on } U\ ,$$
$$u \;=\; g \quad \text{on}\,\partial U\ .$$

The weak sense of the PDE here means that, for any $v \in \mathcal{C}_0^\infty(U)$

$$\int_U \sum_{i=1}^n L_{p_i}(Du,u,x)v_{x_i} + L_z(Du,u,x)v\,dx \;=\; 0\ ,$$

and we already derived this formula in our computation of the first variation of $I[u]$ with respect to the variation $v$.

Step III. Prove the *Regularity Theorem* that any $W^{1,q}$ weak solution of this Dirichlet problem is actually smooth and lies in the original class $\mathcal{A}$. This final step is the heart of *elliptic regularity theory* which is a big subject. Later we will prove some results under additional restrictions on the form of $L$.

We now outline Step I while mentioning the key properties of Sobolev spaces that we need.

(1). Show there exists at least one $w \in \mathcal{A}$. This is not a difficult construction using the function $x \mapsto \operatorname{dist}(x, \partial U)$.

(2). Assume $L$ satisfies a *coercivity* condition: $L(p,z,x) \geq \alpha|p|^q - \beta$ which will imply that

$$\inf_{u\in W^{1,q}(U)} I[u] \;>\; -\infty\ ,$$

and that any sequence $u_k \in W^{1,q}(U)$ with $u_k|\partial U = g$ and $I[u_k] \to \inf_{\mathcal{A}} I$ automatically has

$$\sup_k \int_U |Du_k|^q\,dx \;<\; \infty\ .$$

(3). The functions $v_k = u_k - w$ then belong to $W_0^{1,q}(U)$.

(4). The *Sobolev Inequality* for $v \in W_0^{1,q}(U)$ states that

$$\Big(\int_U |v|^{q^*}\,dx\Big)^{1/q^*} \;\le\; C\Big(\int_U |v|^q\,dx\Big)^{1/q}$$

where $q^* = \frac{nq}{n-q}$. This, along with Hölder's inequality,

$$\Big(\int_U |v|^q\,dx\Big)^{1/q} \;\le\; C(U,q)\Big(\int_U |v|^{q^*}\,dx\Big)^{1/q^*} \;,$$

(since $q < q^*$) implies that $\sup_k \int_U |v_k|^q\,dx \;<\; \infty$.

(5). With the bounds from (2) and (4), we can use the *Sobolev Weak Compactness Theorem* to guarantee that a subsequence $v_{k'}$ converges weakly in $W^{1,q}$ to some function $v \in W^{1,q}(U)$. Also $v$ belongs to $W_0^{1,q}(U)$ by *Sobolev Trace Theory.* It follows that $u_k$ converges weakly in $W^{1,q}$ to $u \equiv v + w \in W^{1,q}(U)$ and that $u = g$ on $\partial U$ in the trace sense.

(6). Suppose now that $p \mapsto L(p,z,x)$ is *convex*, which is equivalent to the pointwise condititon that

$$\sum_{i,j=1}^{n} L_{p_i,p_j}(p,u,x)\xi_i\xi_j \;\ge\; 0 \;\text{ for all }\; \xi = (\xi_1,\cdots,\xi_n) \in \mathbf{R}^n \;.$$

This implies that, under this $W^{1,q}$ weak convergence, we have the *lower semi-continuity*

$$I[u] \;\le\; \liminf_{k'\to\infty} I[u_{k'}] \;=\; \inf_{\mathcal{A}} I \;.$$

We conclude that $u$ is an $I$ minimizer among $W^{1,q}(U)$ functions having trace $g$ on $\partial U$. This completes Step I.