Existence of a Minimizer with Dirichlet Boundary Data

Then initial goal in the course is to describe, for a given smooth function \( g \) on the boundary of a smoothly bounded domain \( U \subseteq \mathbb{R}^n \), how to obtain in the admissible class
\[
\mathcal{A} = \{ u \in C^\infty(U) : u|\partial U = g \} ,
\]
a minimizer for a functional \( I[u] = \int_U L(Du, u, x) \, dx \) for certain smooth integrands \( L : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \). This is not true for all integrands, and we will need to impose further conditions (in (2) and (6) below) on \( L \). Also this existence problem is usually solved in 3 steps:

Step I. Prove the existence of a minimizer \( u \) in a larger admissible class of \( W^{1,q}(U) \) Sobolev functions (with \( q < n \) depending on \( L \)).

Step II. Showing that \( u \) satisfies (in a weak sense) the Euler-Lagrange Dirichlet problem
\[
- \sum_{i=1}^{n} (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) = 0 \quad \text{on} \ U ,
\]
\[
u = g \quad \text{on} \ \partial U .
\]
The weak sense of the PDE here means, for any \( v \in C^\infty_0(U) \)
\[
\int_U \sum_{i=1}^{n} L_{p_i}(Du, u, x)v_{x_i} + L_z(Du, u, x)v \, dx = 0 ,
\]
and we already derived this formula in our computation of the first variation of \( I[u] \) with respect to the variation \( v \).

Step III. Prove the Regularity Theorem that any \( W^{1,q} \) weak solution of this Dirichlet problem is actually smooth and lies in the original class \( \mathcal{A} \). This final step is the heart of elliptic regularity theory which is a big subject. Later we will prove some results under additional restrictions on the form of \( L \).

We now outline Step I while mentioning the key properties of Sobolev spaces that we need.

(1). Show there exists at least one \( w \in \mathcal{A} \). This is not a difficult construction using the function \( x \mapsto \operatorname{dist}(x, \partial U) \).

(2). Assume \( L \) satisfies a coercivity condition: \( L(p, z, x) \geq \alpha|p|^q - \beta \) which will imply that
\[
\inf_{u \in W^{1,q}(U)} I[u] > -\infty ,
\]
and that any sequence \( u_k \in W^{1,q}(U) \) with \( u_k|\partial U = g \) and \( I[u_k] \rightarrow \inf \mathcal{A} I \) automatically has
\[
\sup_k \int_U |Du_k|^q \, dx < \infty .
\]
(3). The functions \( v_k = u_k - w \) then belong to \( W^{1,q}_0(U) \).

(4). The \textit{Sobolev Inequality} for \( v \in W^{1,q}_0(U) \) states that
\[
\left( \int_U |v|^{q^*} \, dx \right)^{1/q^*} \leq C \left( \int_U |v|^q \, dx \right)^{1/q}
\]
where \( q^* = \frac{nq}{n-q} \). This, along with Hölder’s inequality,
\[
\left( \int_U |v|^q \, dx \right)^{1/q} \leq C(U,q) \left( \int_U |v|^{q^*} \, dx \right)^{1/q^*},
\]
(since \( q < q^* \)) implies that \( \sup_k \int_U |v_k|^q \, dx < \infty \).

(5). With the bounds from (2) and (4), we can use the \textit{Sobolev Weak Compactness Theorem} to guarantee that a subsequence \( v_{k'} \) converges weakly in \( W^{1,q} \) to some function \( v \in W^{1,q}(U) \). Also \( v \) belongs to \( W^{1,q}_0(U) \) by \textit{Sobolev Trace Theory}. It follows that \( u_k \) converges weakly in \( W^{1,q} \) to \( u \equiv v + w \in W^{1,q}(U) \) and that \( u = g \) on \( \partial U \) in the trace sense.

(6). Suppose now that \( p \mapsto L(p,z,x) \) is \textit{convex}, which is equivalent to the pointwise condition that
\[
\sum_{i,j=1}^n L_{p_i,p_j}(p,u,x)\xi_i\xi_j \geq 0 \quad \text{for all } \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.
\]
This implies that, under this \( W^{1,q} \) weak convergence, we have the \textit{lower semi-continuity}
\[
I[u] \leq \liminf_{k' \to \infty} I[u_{k'}] = \inf \mathcal{A} I.
\]
We conclude that \( u \) is an \( I \) minimizer among \( W^{1,q}(U) \) functions having trace \( g \) on \( \partial U \). This completes Step I.