Homework 9 due Wed. Mar.21

1. Verify by direct computation that, for $n \geq 3$, the function $u : \mathbf{R}^n \to \mathbf{S}^{n-1}$ given by $u(x) = x/|x|$ satisfies the harmonic map equation $\Delta u + |Du|^2 u = 0$.

2. Consider a solution u of the obstacle problem discussed in section 8.4.2 with $n = 2$, and assume that the boundary between the contact set C and the non-contact set O is locally a smooth curve Γ and that on $\bar{O}$ the solution is continuously differentiable all the way up to Γ. Show that above Γ the graph of u is tangent to the graph of h.

3. Suppose U is the open unit ball in $\mathbf{R}^2$, $h : \mathbf{R}^2 \to \mathbf{R}$, $h(x, y) = 1 - 2(x^2 + y^2)$ and

$$\mathcal{A} = \{w \in W_0^{1,2}(U) \ : \ w \geq h(x, y) \text{ for } (x, y) \in U\}$$

(a) Show that $\mathcal{A} \neq \emptyset$.

(b) Guess a formula for the minimizer of $\int_U |Du|^2 \, dxdy$ in $\mathcal{A}$. Hint: Assume $u = u(r)$ where $r = \sqrt{x^2 + y^2}$ and use #2.