Mid-Term Exam Solutions, Math 425, Fall, 2005

1. The family

$$\mathcal{A} \;=\; \{\, A \subset \mathbf{N} : \text{ either } A \text{ is finite or } \mathbf{N} \setminus A \text{ is finite} \,\}$$

is countable. To see this, note that $A = \cup_{k=0}^{\infty} \mathcal{A}_k \cup \mathcal{B}_k$ where

$$\mathcal{A}_k \;=\; \{\, A \subset \{1, 2, \cdots, k\} \,\}$$

has a finite number (2^k) of elements and

$$\mathcal{B}_k \;=\; \{\, \mathbf{R}^n \setminus A \;:\; A \in \mathcal{A}_k \,\}$$

also has only a finite number of elements. A countable union of finite sets is countable.

2. There does exist a compact subset K of irrational numbers with positive 1-dimensional Lebesgue measure. One can take $K = [0,4] \setminus \cup_{i=1}^{\infty}(r_i - 2^{-i}, r_i + 2^{-i})$ where $\{r_1, r_2, \cdots\} = \mathbf{Q}$, the rational numbers. Here $\lambda(K) \geq 4 - \sum_{i=1}^{\infty} 2^{-i+1} = 2$.

3. (a) For any subset A of $\mathbf{R}^n$, $\lambda^*(A) \;=\; \lambda(\cap_{i=1}^{\infty} G_i)$ for some open sets $G_1, G_2, \cdots$ containing A. In case $\lambda^*(A) = \infty$, simply take $G_i = \mathbf{R}^n$. For $\lambda^*(A) < \infty$, choose open $H_j \supset A$ with $\lambda(H_j) < \lambda^*(A) + 1/j$. Letting $G_i = \cap_{j=1}^{j} H_j$ we see that $G_1 \supset G_2 \supset \cdots$, $\lambda(G_1) < \infty$, and

$$\lambda^*(A) \;\leq\; \lambda(\cap_{i=1}^{\infty} G_i) \;=\; \lim_{i \to \infty} \lambda(G_i) \;\leq\; \lim_{i \to \infty} \lambda(H_i) \;\leq\; \lambda^*(A)\ .$$

(b) Suppose $Z \;=\; \cap_{i=1}^{\infty} G_i \;\setminus\; A$. If $\lambda^*(Z) \;=\; 0$, then Z is measurable and hence the difference $A \;=\; (\cap_{i=1}^{\infty} G_i) \setminus Z$ is also measurable. Conversely, if A is measurable, then $\lambda(Z) = \lambda(\cap_{i=1}^{\infty} G_i) - \lambda(A) = 0$ because $\lambda(\cap_{i=1}^{\infty} G_i) = \lambda(A) < \infty$.

4. If $A \subset \mathbf{R}^n$, $B \subset \mathbf{R}^n$, and $\delta \;=\; \operatorname{dist}(A, B) > 0$, then $U = \{x \;:\; \operatorname{dist}(x, A) < \delta/2\}$ and $V = \{x \;:\; \operatorname{dist}(x, B) < \delta/2\}$ are disjoint open sets. Choosing, for $\varepsilon > 0$ and an open set G so that $\lambda(G) \leq \lambda^*(A \cup B) + \varepsilon$, we conclude that

$$\lambda^*(A) + \lambda_*(B) \;\leq\; \lambda^*(A) + \lambda^*(B) \;\leq\; \lambda(G \cap U) + \lambda(G \cap V) \;\leq\; \lambda(G) \;\leq\; \lambda^*(A \cup B) + \varepsilon$$

and then let $\varepsilon \to 0$.

5. If $g : [0,1] \to \mathbf{R}^2$ satisfies $|g(s) - g(t)| \leq |s - t|$ for all $0 \leq s \leq t \leq 1$, then the image

$$g([0,1]) \;=\; \{g(t) \;:\; t \in [0,1]\}$$

has 2-dimensional Lebesgue measure zero. In fact, the map $f \;:\; [0,1] \times \mathbf{R} \;\to\; \mathbf{R}^2$, $f(x,y) = g(x)$ has $\operatorname{Lip} f = \operatorname{Lip} g = 1$ so that

$$\lambda_2\big(g([0,1])\big) \;\leq\; \lambda_2\big(f([0,1] \times \{0\})\big) \leq\; (\operatorname{Lip} f)^2 \lambda_2\big([0,1] \times \{0\}\big) \;=\; 0\ .$$

Or more directly, note that for each positive integer k

$$g([0,1]) \;\subset\; \cup_{j=1}^{k} g([\frac{j-1}{k}, \frac{j}{k}]) \;\subset\; \bigcup_{j=1}^{k} \overline{B\big(g(\frac{j}{k}), \frac{1}{k}\big)}$$

so that $\lambda_2\big(g([0,1])\big) \leq k \cdot \pi(\frac{1}{k})^2 \;\to\; 0$ as $k \to \infty$.

6. Suppose $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are Lebesgue measurable and $F : \mathbf{R}^2 \to \mathbf{R}$ is continuous. The composition $h : \mathbf{R} \to \mathbf{R}$, $h(x) = F\big(f(x), g(x)\big)$ is measurable. In fact, for any $t \in \mathbf{R}$, $G = F^{-1}\big((t, \infty)\big)$ is open in $\mathbf{R}^2$. By Problem 3, $G = \cup_{i=1}^{\infty}[a_i, b_i] \times [c_i, d_i]$ for some real numbers a_i, b_i, c_i, d_i. It follows that the set

$$h^{-1}\big((t,\infty)\big) = (f,g)^{-1}(G) = \cup_{i=1}^{\infty}(f,g)^{-1}([a_i,b_i]\times[c_i,d_i]) \ = \ \cup_{i=1}^{\infty}\big(f^{-1}[a_i,b_i]\cap g^{-1}[c_i,d_i]\big)$$

is measurable.

7. To find the Lebesgue measure of $g(B)$ where B is the open unit ball in $\mathbf{R}^2$ and $g(x, y) = (x + 2y + 3, x - y - 4)$, we first note, by the translation invariance of Lebesgue measure, that $g(b) = h(B)$ where $h(x, y) = (x + 2y, x - y)$. Next we note that $h([0, 1]\times[0, 1])$ is the parallelogram with vertices $(0, 0), (1, 1), (3, 0), (2, -1)$, and the area of this parallelogram is 3. (The area formula $A = bh$ for a parallelogram is easily checked using the translation and orthogonal invariance of 2 dimensional Lebesgue measure.) It follows, by linearity and translation invariance, that $\lambda\big(h(Q)\big) = 3\lambda(Q)$ for any cube $Q \subset \mathbf{R}^2$. Of course, both the boundary of Q and the boundary of $h(Q)$, being contained in 4 lines, has 2 dimensional Lebesgue measure zero. Using Problem 3 and the fact that h is injective, we conclude that $\lambda\big(h(B)\big) = 3\lambda(B) = 3\pi$.

8. Suppose $0 < \alpha < \infty$. A map $f : \mathbf{R}^n \to \mathbf{R}^n$ is called an α *similarity* if $f(x) - f(0)$ is linear and $|f(x) - f(y)| = \alpha|x - y|$ for all $x, y \in \mathbf{R}^n$.

(a) Show that $\lambda\big(f(A)\big) = \alpha^n\lambda(A)$ for any α similarity f and any Lebesgue measurable subset A of $\mathbf{R}^n$. Since $\operatorname{Lip} f \le \alpha$

$$\lambda\big(f(A)\big) = \alpha^n\lambda(A) \ .$$

The α similarity definition also implies that f is injective with $\operatorname{Lip} f^{-1} \le 1/\alpha$ so we obtain the opposite inequality

$$\lambda(A) \ \le \ \operatorname{Lip} f^{-1}\lambda\big(f(A)\big) \ \le \ (\frac{1}{\alpha})^n\lambda\big(f(A)\big) \ .$$

(b) Suppose that E is a bounded measurable set and that

$$E \ = \ f_1(E) \cup f_2(E) \cup \cdots \cup f_m(E)$$

for some α similarities $f_1, f_2, \cdots, f_m$ of $\mathbf{R}^n$ such that $f_i(E)\cap f_j(E) = \emptyset$ for $1 \le i < j \le m$. From (a) and the measurability of E and of each $f_i(E)$, we get the equation

$$\lambda(E) \ = \ \lambda\big(f_1(E)\big) + \cdots + \lambda\big(f_m(E)\big) \ = \ m\alpha^n\lambda(E) \ . \qquad (*)$$

Here $\lambda(E) < \infty$ because E is bounded. If $\lambda(E) > 0$, then we can cancel it to get the desired equation $\alpha = (1/m)^{1/n}$.

(c) An example with $\lambda(E) = 0$, $m = 2$ and $\alpha = 1/2$ is the "interval" $E = [0, 2) \times \{0\}$ with the 2 similarities

$$f_1(x, y) = \frac{1}{2}(x, y),\ f_2(x, y) = \frac{1}{2}(x + 2, y)\ .$$

An example with $\lambda(E) > 0$, $m = 4$, and $\alpha = (1/4)^{1/2} = 1/2$ is the "square" $E = [0, 2) \times [0, 2)$ with the 4 similarities

$$f_1(x, y) = \frac{1}{2}(x, y),\ f_2(x, y) = \frac{1}{2}(x+2, y),\ f_3(x, y) = \frac{1}{2}(x, y+2),\ f_4(x, y) = \frac{1}{2}(x+2, y+2)\ .$$

A more interesting example with $\lambda(E) = 0$, $m = 4$, and $\alpha = \frac{1}{3}$ is $E = C \times C$ where C is the standard terniary Cantor set and

$$f_1(x, y) = \frac{1}{3}(x, y),\ f_2(x, y) = \frac{1}{3}(x+2, y),\ f_3(x, y) = \frac{1}{3}(x, y+2),\ f_4(x, y) = \frac{1}{3}(x+2, y+2)\ .$$

[Here equation (*) just becomes $0 = 0$, but a more useful equation here occurs with $m = 4$, $\alpha = \frac{1}{3}$, n replaced by $\beta = \log 4/\log 3$, and n dimensional Lebesgue measure λ replaced by β dimensional "Hausdorff" measure.]

9. Suppose that A and B are Lebesgue measurable subsets of $\mathbf{R}$. To see that $A \times B$ is Lebesgue measurable in $\mathbf{R}^2$ with $\lambda_2(A \times B) = \lambda_1(A)\lambda_1(B)$, we will treat several cases. This is clear in case both A and B are closed intervals. A finite union of closed intervals is a finite disjoint union of closed intervals. It both A and B are such finite disjoint unions, say $A = \cup_{i=1}^{m}[a_i, b_i]$ and $B = \cup_{j=1}^{n}[c_j, d_j]$, then $A \times B$ is the disjoint union of the rectangles $[a_i, b_i] \times [c_j, d_j]$ and one obtains the measurability. The product formula follows by linearity. If both A and B are open sets, then taking sup's over enclosed special polygon's readily gives $\lambda_1(A)\lambda_1(B) \leq \lambda_2(A \times B)$. To verify equality, we observe that I is a special polygon in $A \times B$, then the projection I_X of I onto the X-axis is contained in A and similarly $I_Y \subset B$. So $I \subset I_X \times I_Y \subset A \times B$, and this allows us to get the opposite inequality. For compact sets A, B. where the measure is given by the infema over enclosing open sets, we now get immediately the inequality $\lambda_1(A)\lambda_1(B) \geq \lambda_2(A \times B)$. For the opposite inequality, we observe that for any open set G containing the compact set $A \times B$, there are open sets G_X and G_Y with $A \times B \subset G_X \times G_Y \subset G$. Here we can get G_X and G_Y by taking δ open neighborhoods of A and B where $\sqrt{2}\delta = \text{dist}\,(A \times B, G^c)$. The general case of measurable A and B now follows by approximation.