Homework 1 due Wednesday, August 31.
Chapter 1: #2(b)(c)(e), #3, #9, #11, #18, #24, #32(e) (In each chapter of the textbook, the problems are numbered, and scattered throughout the text. For example, #2(b)(c)(e) refers to parts (b)(c), and (e) of Problem 2 on page 3.)

Homework 2 due Wednesday, Sept.7.
Chapter 1: #47, #53, #59, #60, Chapter 2: #2, #3, #4(a)(b),

Extra Problem: Prove that the interval $[0,1]$ is not homeomorphic to the solid square $[0,1]x[0,1]$. (See page 18. Hint: You may use Problem 45. What happens when we remove 1 point? A warning: there actually is a continuous function mapping the interval onto the whole square, but it is not injective.)

Homework 3 due Wednesday, Sept.14.
Chapter 2: #23, #25.

The purpose of the rest of this week's homework is to show how to get decompositions involving disjoint closed balls (compare Problem #36, Chapter2). This will give us an alternate proof of the rotation invariance of Lebesgue measure.
Problem 3. Show that in $\mathbf{R}^n$, $(2/\sqrt{n})^n \le \lambda\big(\mathbf{B}(0,1)\big) \le 2^n$. (The exact value will be found in Chapter 9.)
Problem 4. Suppose $r > 0$, G is an open subset of $\mathbf{R}^n$, and $rG = \{rx \ : \ x \in G\}$. Show that $\lambda(rG) = r^n\lambda(G)$.
Problem 5. Show that the unit sphere $\partial\mathbf{B}(0,1) = \{x \ : \ |x| = 1\}$ has n dimensional Lebesgue measure zero. Hint: Use #4 with $G = \mathbf{B}(0,1)$.
Problem 6. Show that any open set G equals a countable union of nonoverlapping closed dyadic cubes. Here a closed dyadic cube is a set of the form 2^jQ_k where $j \in \mathbf{Z}$, $k = (k_1,\ldots,k_n) \in \mathbf{Z}^n$, and $Q_k = [k_1, k_1+1] \times \ldots \times [k_n, k_n+1]$.
Problem 7. Show that any open set $G = Z \cup \cup_{j=1}^{\infty} B_j$ where $\lambda(Z) = 0$ and the sets B_j are disjoint closed balls. Hint: Fix a number ρ with $1 - (n)^{-n/2} < \rho < 1$, and use #1, #2, and #4 to show there are finitely many disjoint closed balls $B_1,\ldots,B_{N_1}$ in G so that $G_1 = G \setminus \cup_{j=1}^{N_1} B_j$ has $\lambda(G_1) \le \rho\lambda(G)$. Then repeat the argument to get disjoint closed balls $B_{N_1+1},\ldots,B_{N_2}$ in G_1 so that $G_2 = G_1 \setminus \cup_{j=N_1+1}^{N_2} B_j$ has $\lambda(G_2) \le \rho\lambda(G_1)$, etc.

Homework 4 due Wednesday, Sept.21.

[Changes: added “linear” assumption to #7,#8 and changed the hints in #3,#6.]

Chapter 2: #28, #34.

One purpose of the rest of this week's homework is to show the rotation invariance of Lebesgue measure.
Problem 3. Show that $\lambda(Z) = 0 \iff$ For every $\varepsilon > 0$ there are open balls B_i with $Z \subset \cup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} \lambda(B_i) < \varepsilon$. [Hint: For the implication one can first choose an open

U containing Z with $\lambda(U) < \varepsilon/(2\sqrt{n})^n$, then use Problem 6 from last week and note that any closed cube Q is contained in an open ball B with $\lambda(B) < (2\sqrt{n})^n\lambda(Q)$.]

Problem 4. A function $f : \mathbf{R}^n \to \mathbf{R}^n$ is *Lipschitz* if the number

$$\operatorname{Lip} f \;\equiv\; \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)}$$

is finite. Show that for any open ball B, $\lambda(f(B)) \leq (\operatorname{Lip} f)^n \lambda(B)$. [Hint: Recall Problem 4 from last week.]

Problem 5. Show that $\lambda(f(Z)) = 0$ for any Lipschitz $f : \mathbf{R}^n \to \mathbf{R}^n$ and any set $Z \subset \mathbf{R}^n$ with $\lambda(Z) = 0$.

Problem 6. Show that $\lambda(f(G)) \leq (\operatorname{Lip} f)^n \lambda(G)$ for any open subset G of $\mathbf{R}^n$. [Hint: Use Problem 7 from last week.]

Problem 7. A linear map $g : \mathbf{R}^n \to \mathbf{R}^n$ is *orthogonal* if $g(x) \cdot g(y) = x \cdot y$ for all $x, y \in \mathbf{R}^n$. Show that such a map is injective and surjective with $\operatorname{Lip} g = 1$ and $\operatorname{Lip} g^{-1} = 1$. [Hint: $d(x,y)^2 = |x-y|^2 = x \cdot x - 2x \cdot y + y \cdot y$.

Problem 8.(Rotation Invariance) Show that $\lambda(g(A)) = \lambda(A)$ for any orthogonal linear map $g : \mathbf{R}^n \to \mathbf{R}^n$ and any Lebesgue measurable $A \subset \mathbf{R}^n$.

Homework 5 due Monday, Oct.3 **NOTE THIS NEW DATE BECAUSE OF RITA.**

Chapter 2: #32, #35, #42, #45, #46, #47 [Hint: use#46]. Chapter 4: #2.

Problem 8.(outer measure) Suppose $\mu(E) \in [0,\infty]$ for all $E \subset \mathbf{R}^n$, $\mu(\emptyset) = 0$, and $\mu(\cup_{i=1}^\infty E_i) \leq \sum_{i=1}^\infty \mu(E_i)$ whenever $E_i \subset \mathbf{R}^n$. One says that a subset A of $\mathbf{R}^n$ is μ *measurable* if

$$\mu(E) \;=\; \mu(E \cap A) + \mu(E \setminus A) \quad \text{for all} \quad E \subset \mathbf{R}^n \; .$$

Show that $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$ whenever A_i are disjoint μ measurable subsets of $\mathbf{R}^n$. [Hint: Prove the superadditivity first for 2 disjoint μ measurable sets, then inductively for finitely many, then take a limit.]

Homework 6 due Wed., Oct.12 (**after break**)

Chapter 4: #7,

Chapter 5: #2, #8, #10, #11, #13, #14.

Homework 7 due Wed., Oct.19

Chapter 5: #16, #18, #20, #23.

Chapter 6: #1, #5, #8.

Problem 8. Prove that for every measurable simple function s on $\mathbf{R}^n$ there exists a sequence of continuous functions $f_i : \mathbf{R}^n \to \mathbf{R}$ so that

$$\lim_{i \to \infty} f_i(x) \;=\; s(x) \quad \text{for almost every} \quad x \in \mathbf{R}^n \; .$$

[Hint: See the Theorem in Chapter 1, Section F.]

Homework 8 due Mon., Oct.31

Chapter 6: #12, #19, #21, #22, #24, #36,
Problem 7. Suppose that $f \in L^1(E)$ where E is a measurable set of finite positive measure. Show that there is a point $x \in E$ with $|f(x)| \le (\lambda(E)^{-1} \int_E |f|\, d\lambda$.
Problem 8. (Corrected!) Suppose $f, f_1, f_2, f_3, \cdots$ are all L^1 functions on $\mathbf{R}^n$ and that $\int |f_k - f|\, d\lambda \to 0$ as $k \to \infty$. Prove that for every $\varepsilon > 0$ there exists an integer k_ε so that

$$\lambda\{x \;: |f_k(x) - f(x)| > \varepsilon\} \;<\; \varepsilon$$

whenever $k \ge k_\varepsilon$.

Homework 9 due Mon., Nov.7

Chapter 6: #28, #30, #35, #40, #44, #45,
Problem 7. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. For $E \subset X$, we define

$$\nu(E) \;=\; \inf\{\mu(A) \;:\; E \subset A \in \mathcal{M}\}.$$

Prove that ν is an outer measure on X (i.e. countably subadditive) and that one has the equality

$$\nu(E) \;=\; \nu(E \cap A) \;+\; \nu(E \setminus A) \quad \text{for all } \; A \in \mathcal{M} \;.$$

(Thus every measurable set for μ is also measurable for ν.) [Hint you may use some results from the book for Lebesgue measure if you just check that the proofs carry over to the general measure μ.)

Homework 10 due Mon., Nov.14

Chapter 6: #32, #36, #39, #42, Chapter 7: #22

Problem 6. (Corrected version: I changed a 2 to $\frac{1}{2}$ in the definition of f_j) For each positive integer j, let

$$f_j \;=\; \frac{1}{2} j \cdot \chi_{[\,-1/j, 1/j]} \;\; \text{and} \;\; \nu_j(E) = \int_E f_j\, d\lambda \;\; \text{for Lebesgue measurable } \; E \subset \mathbf{R} \;.$$

(1) Show that the measures ν_j approach δ_0 as $j \to \infty$ in the sense that

$$\int g\, d\nu_j \;\to\; \int g\, d\delta_0 \;\; \text{for every continuous } \; g : \mathbf{R} \to \mathbf{R} \;.$$

(2) Is it true that $\nu_j(E) \to \delta_0(E)$ for all Borel subsets E of $\mathbf{R}$? Prove this or find an example of a Borel E where this isn't true.

Homework 10 due Mon., Nov.21 (with 2 corrections. $B \to A$ in Problem 2(2) and $|b_i - a_i| \leq \delta$ added in the definition in Problem 3.)

Chapter 7: #20.

Problem 2. For any 2 Lebesgue measurable subsets $A, B \subset \mathbf{R}^n$, define

$$\text{dist}_\lambda(A, B) \;=\; \lambda(A \setminus B) + \lambda(B \setminus A) \;.$$

(1) Show that the $\text{dist}_\lambda(A, C) \;\leq \text{dist}_\lambda(A, B) \;+ \text{dist}_\lambda(B, C)$.

(2) Prove that $\lim_{c \to 0} \text{dist}_\lambda(A + c, A) = 0$ where $A + c = \{a + c \,:\, a \in A\}$.

Problem 3. For $0 \leq r < \infty$ and $A \subset \mathbf{R}$, we define the Hausdorff outer measure

$$\mathcal{H}^r(A) \;=\; \lim_{\delta \to 0} \inf\{\, \sum_{i=1}^{\infty} (b_i - a_i)^r \;:\; A \subset \cup_{i=1}^{\infty} [a_i, b_i]\,,\; |b_i - a_i| \leq \delta\} \;.$$

(1) Prove that if $\mathcal{H}^r(A) < \infty$, then $\mathcal{H}^s(A) = 0$ for all $s > r$.

(2) Prove that if C is the standard tertiary Cantor set, then $\mathcal{H}^t(C) < \infty$ where $t = \log 2 / \log 3$.

Homework 11 due Wed., Nov.30

Chapter 8: #1, #2, #7, #14, #16, #19

Chapter 10: #1, #21 [Hint: Use Hölder's inequality.]