Algebraic Riemann surfaces

General Problem from Algebraic Geometry

Describe $V = \{(z,w) \in \mathbf{C} \times \mathbf{C} \ : \ P(z,w) = 0\}$ where P is a (not identically zero) polynomial in 2 variables.

We will show how V is a finite union of Riemann surfaces.

First we may (after swapping z and w if necessary) write

$$P(z,w) \ = \ b_0(z)w^n + b_1(z)w^{n-1} + \ldots + b_n(z) \ = \ b_0(z)\Phi(z,w)$$

where $b_0(z) \not\equiv 0$ and

$$\Phi(z,w) \ = \ w^n + a_1(z)w^{n-1} + \ldots + a_n(z)$$

where $a_i(z) = \frac{b_i(z)}{b_0(z)}$ are rational functions. If $b_0^{-1}\{0\} = \{\alpha_1, \ldots \alpha_k\}$, then

$$V \ = \ \big(\cup_{i=1}^k \{a_i\} \times \mathbf{C} \big) \cup \Phi^{-1}\{0\big) \ .$$

To describe $\Phi^{-1}\{0\big)$, we let $Z = b_0^{-1}\{0\}$ and $\rho_i(z)$ for $z \in \mathbf{C} \setminus Z$ and $i = 1, \ldots, n$, denote the roots, with possible repetitions, of $\Phi(z,w) = 0$. The discriminant (with respect to the w variable) of $\Phi(z,w)$ is given by

$$D(z) \ = \ \prod_{i \neq j} \big(\rho_i(z) - \rho_j(z)\big)^2 \ .$$

The relevant algebraic fact we wish to use is (see Lang, *Algebra*, V,§9-10):

$$D \equiv 0 \ \text{ iff } \ \Phi \text{ has a repeated root } \ \text{ iff } \ \Phi \text{ has a repeated factor } .$$

For example, $w^2 - z$ has $D^{-1}\{0\}$ equalling the single point $\{0\}$ while $w^4 - 2w^2z + z^4 = (w^2 - z)^2$, which has the same zero set, has discriminant identically 0.

Without changing $V = \Phi^{-1}\{0\}$, we may now assume Φ has no repeated factors and decompose $\Phi = \Pi_{j=1}^J \Phi_j$ where the Φ_j are irreducible and distinct. Since $\Phi^{-1}\{0\} = \cup_{j=1}^J \Phi_j^{-1}\{0\}$, we may, by replacing Φ by each Φ_j now assume

$$\Phi \text{ itself is irreducible and hence } \ D \not\equiv 0 \ .$$

To show how V is a Riemann surface we will show that $\Pi : V \to \mathbf{C} \setminus Z$, $\Pi(z,w) = z$, *is a holomorphic branched covering map with branch image set* $D^{-1}\{0\}$.

A key tool is the construction of a product neighborhood $\mathbf{B}_\delta(a) \times \mathbf{B}_\epsilon(b)$ about any point $(a,b) \in \mathbf{C}^2$. First, since $\Phi(a,\cdot)$ has only finitely many zeroes, we find that, for all sufficiently small positive ϵ, $\Phi(a,w) \neq 0$ whenever $|w-b| = \epsilon$. Then, for all sufficiently small positive δ depending on ϵ, one still has $\Phi(z,w) \neq 0$ whenever $|z-a| < \delta$ and $|w-b| = \epsilon$. Thus Φ does not vanish on the set $\mathbf{B}_\delta(a) \times \partial\mathbf{B}_\epsilon(b)$. Now we can use a formula from complex analysis (Ahlfors, *Complex Analysis*, 3.3, Th.10) to see that the number of solutions of $\Phi(z,w) = 0$ with $|z-a| < \delta$ and $|w-b| < \epsilon$ is

$$N(z) \;=\; \frac{1}{2\pi i}\int_{\partial\mathbf{B}_\epsilon(b)} \frac{\frac{\partial\Phi}{\partial w}(z,\zeta)}{\Phi(z,\zeta)}\,d\zeta\ ,$$

which is continuous, integer-valued, and hence the constant $N(a)$. For $D(a) \neq 0$ and $(a,b) \in V$, $N(a) = 1$ and the arbitrariness of ϵ shows that the corresponding root $w = \rho(z)$ of $\Phi(z,w) = 0$ in $\mathbf{B}_\epsilon(b)$ is continuous on $\mathbf{B}_\delta(a)$. Moreover, the formula (see Ahlfors, §5.3 eqn(44))

$$\rho(z) \;=\; \frac{1}{2\pi i}\int_{\partial\mathbf{B}_\epsilon(b)} \zeta\frac{\frac{\partial\Phi}{\partial w}(z,\zeta)}{\Phi(z,\zeta)}\,d\zeta$$

shows that ρ is also holomorphic on $\mathbf{B}_\delta(a)$. Thus $\Pi\big|\Phi^{-1}\{0\} \setminus (D\circ\Pi)^{-1}\{0\}$is an n sheeted holomorphic covering map.

For each positive integer k we may also apply the formula (Ahlfors, §5.3 eqn(44)) to the function w^k to conclude that the power sum

$$\sum_{j=1}^{n}\rho_i(z)^k \;=\; \frac{1}{2\pi i}\int_{\partial\mathbf{B}_\epsilon(b)} \zeta^k\frac{\frac{\partial\Phi}{\partial w}(z,\zeta)}{\Phi(z,\zeta)}\,d\zeta$$

is a holomorphic function on $\mathbf{C}\setminus Z$. *Any* symmetric polynomial in the $\rho_i(z)$'s is a polynomial in the power sums for various k, and hence is also holomorphic. Moreover, since it is easy to estimate the root of a monic polynomial in terms of its coefficients, we see that this holomorphic function has polynomial growth in z as $|z| \to \infty$ or in $\frac{1}{|z-a|}$ as $z \to a \in Z$, hence is a rational function. In particular, the discriminant $D(z)$ is a rational function of z, and $D^{-1}\{0\}$ is a finite set.

Next consider a point $a \in \mathbf{C}\setminus Z$ with $D(a) = 0$. Lifting a small circle $\partial\mathbf{B}_\delta(a)$ gives a finite number of circles $C_1,\ldots,C_j$ which Π maps down with some positive integer multiplicities $k_1,k_2,\ldots,k_j$ where $k_1+k_2+\ldots+k_j = n$. Our formula for $N(z)$ shows that each such C_i shrinks to the point (a,b) as $\delta \downarrow 0$. It determines a component V_i of $V\cap\Pi^{-1}\big(\mathbf{B}_\delta(a)\setminus\{a\}\big)$, which is a punctured disk. Then $\big[\Pi(\cdot)-a\big]^{1/k_i}$

defines a holomorphic coordinate for $V_i \cup \{(a,b)\}$. We now conclude that $\Phi^{-1}\{0\}$ is a finite union of Riemann surfaces.

Finally we can lift a small circle $\partial \mathbf{B}_\delta(a)$ for $a \in Z$ or a large circle $\partial \mathbf{B}_R(0)$, large enough to contain $D^{-1}\{0\}$ in its interior, to see similar local topological behavior in a neighborhood of $a \in Z$ or of ∞. Because of the polynomial growth described above, we similarly obtain, after adding a finite set of points to V, holomorphic coordinates near these points at infinity. We finally obtain a finite union of compact Riemann surfaces.

Remark. A concrete way of obtaining these extra points is to take the closure of V in the complex projective space $\mathbf{C}P^2$ under a usual embedding $\mathbf{C}^2$ in $\mathbf{C}P^2$.

Now we turn to a converse fact:

Riemann's Theorem. *Every compact connected Riemann surface M is biholomorphic to an algebraic Riemann surface as above.*

Proof : We will construct an irreducible $\Phi(v,w)$ and biholomorphic map from $M \setminus \{\text{finite set}\}$ onto $\Phi^{-1}\{0\} \subset \mathbf{C} \times \mathbf{C}$.

First we choose a nonconstant meromorphic function $z = f(P)$ on M. Then f gives a holomorphic branched covering map onto $\hat{\mathbf{C}}$ with a finite number n of sheets. Let B be the image of the branch points and fix a point $z_0 \in \mathbf{C} \setminus B$ so that $f^{-1}\{z_0\}$ equals n points $P_1^0, \ldots, P_n^0$.

Second, we find a meromorphic function g on M so that $g(P_1^0), \ldots, g(P_n^0)$ are distinct complex numbers. To do this we may, for example, let ω_j be a meromorphic 1 form on M with a single order 2 pole at P_j^0 and principal part $-\frac{dz}{(z-z_0)^2}$ in terms of a local parameter $z = f(P)$. Let $c_1, \ldots, c_n$ be distinct complex numbers and

$$g \; = \; (z-z_0)^2 \big(c_1 \frac{\omega_1}{dz} \; + \; \ldots \; + \; c_n \frac{\omega_n}{dz}\big) \; .$$

Then g works because $\frac{(z-z_0)^2 \omega_i}{dz}$ equals 1 at P_i^0 and vanishes at P_j^0 for $j \neq i$.

Now we can construct $\Phi(z,w)$. Letting $f^{-1}\{z\} = \{P_1(z), \ldots, P_n(z)\}$, we use the elementary symmetric functions

$$\begin{aligned}
a_1 &= a_1(z) = -g(P_1) - g(P_2) - \ldots - g(P_n) \\
a_2 &= a_2(z) = g(P_1)g(P_2) + g(P_1)g(P_3) + \ldots = \sum_{i<j} g(P_i)g(P_j) \\
&\;\;\cdot \\
&\;\;\cdot \\
&\;\;\cdot \\
a_n &= a_n(z) = (-1)^n g(P_1) \ldots g(P_n) \; .
\end{aligned}$$

Arguing as before, we find that these functions are by symmetry and growth considerations rational on **C**. Moreover,

$$\Phi(z, w) \;=\; \prod_{i=1}^{n} \big(w - g(P_i(z))\big) \;=\; w^n + a_1(z)w^{n-1} + \ldots + a_n(z) \;,$$

and Φ is irreducible because the complex numbers $g(P_1^0), \ldots, g(P_n^0)$ are distinct.

The map $F = (f, g)$ clearly maps holomorphically onto $\Phi^{-1}\{0\}$. To see that F is one-one, note that for any $P \in M$ with $F(P) = (z_0, w)$, $f(P) = z_0$, $P = P_i^0$ for some i and $w = g(P_i^0)$, hence $F^{-1}\{(z_0, w)\}$ has only one point. Thus F has degree 1 and, being holomorphic between Riemann surfaces, must be one-one. ■