Assignment 6, due 10/29

Let $\mathcal{C}_0^\infty(\mathbf{R}) = \{$infinitely differentiable $f : \mathbf{R} \to \mathbf{R}$ of compact support$\}$.

The *weak derivative* of a locally integrable function $f : \mathbf{R} \to \mathbf{R}$ is the linear function $D_f : \mathcal{C}_0^\infty(\mathbf{R}) \to \mathbf{R}$ defined by

$$D_f(g) \;=\; -\int f g' \, dx \;\text{ for all }\; g \in \mathcal{C}_0^\infty(\mathbf{R}) \;.$$

Ex. 1 Find another formula for $D_f(g)$ in case f is continuously differentiable.

Ex. 2 Find another formula for $D_A(g)$ where $A(x) = |x|$.

Ex. 3 Find another formula for $D_H(g)$ where $H(x) = 0$ for $x \le 0$ and $H(x) = 1$ for $x > 0$.

Ex. 4 Let $\gamma(x,y) = \lambda(|x-y|)H(y-x)$ where $\lambda(r)$ is a cutoff function as on P.297 of Royden's paper. Let

$$(T\phi)(y) \;=\; \int \phi(x)\gamma(x,y)\, dx \text{ for }\; \phi \in \mathcal{C}_0^\infty(\mathbf{R}) \;.$$

Arguing now as in proof of Lemma 1 in Royden's paper, show that $T\phi$ is differentiable and (using Ex.3) that

$$(T\phi)'(y) \;=\; \phi(y) \;+\; \int_{-\infty}^{y} \phi(x)\lambda'(y-x)\, dx \;=\phi(y) \;-\; \int \phi(x)\gamma_x(x,y)\, dx \;.$$

Hint: You may use the fact that $G(y) = \int_a^y F(x,y)\, dx$ is smooth whenever F is smooth and that $G'(y) = F(y,y) + \int_a^y F_y(x,y)\, dx$. This fact follows by applying the Fundamental Theorem of Calculus and the chain rule to differentiate $G(y) = H(y,y)$ where $H(y,z) = \int_a^y F(x,z)\, dx$.

Ex. 5 Using Ex.4 and arguing as in the proof of Lemma 1, show:

If $D_f \in \mathcal{C}^\infty(\mathbf{R})$

(that is, for some $h \in \mathcal{C}^\infty(\mathbf{R})$, $D_f(g) = \int g \cdot h \, dx$ for all $g \in \mathcal{C}_0^\infty(\mathbf{R})$,

then $f \in \mathcal{C}^\infty(\mathbf{R})$.

Hint: Take $g = T\phi$, change variables to obtain the common factor $\phi(x)\, dx$ in all integrals, and use Fubini's Theorem to find the equation

$$-\int_x^\infty \lambda(y-x)h(y)\, dy \;=\; f(x) + \int_x^\infty f(y)\lambda'(y-x)\, dy \;,$$

which shows the smoothness of f.