Math 428 - Solutions to Final Exam, Fall 1998

**1.** The formula $f(z) = \frac{z^3(z-2)}{z^2+4}$ defines a holomorphic map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$.

(a) Find all the zeroes, poles, and branch points of $f$.

zeroes: 0 (with order 3), 2 (with order 1)

poles: $2i$ (with order 1), $-2i$ (with order 1), $\infty$ (with order 2).

branch points: $f$ has branching order 1 at $\infty$. To find the others in $\mathbf{C}$, one solves $f'(z) = 0$. This has a factor $z^2$ corresponding to branching order 2 at 0. After dividing the equation by $z^2$, one obtains the cubic $z^3 - z^2 + 8z - 12 = 0$ which has 3 simple roots (1 real, 2 complex conjugates).

(b) Here $g = \gamma = 0$ because the domain and the range are the Riemann sphere. The covering multiplicity is $n = 4$, the degree of $f$. and the total branching order is $B = 1 + 2 + 3 = 6$. So $g = n(\gamma - 1) + 1 + \frac{B}{2}$ is true.

**2.** For each of the following explain whether or not there exists Riemann surfaces $M$ and $N$ and a nonconstant holomorphic map $f : M \to N$ when

(a) $M$ is a sphere and $N$ is a torus. The map doesn't exist, because Riemann-Hurwitz would imply that $0 = n(1-1)+1+\frac{B}{2}$, contradicting that $B$ is nonnegative and even.

(b) $M$ is a torus and $N$ is a sphere. The map does exist. In Assignment 2 we found a nonconstant meromorphic function on the torus.

(c) $M$ is a torus, $N$ is a torus, and $\deg f = 3$. The map does exist. With $M$ being $\{(z, w) \in \mathcal{C}^2 \ : \ |z| = |w| = 1\}$ with the *standard* conformal structure and $f(e^{i\theta}, e^{i\phi}) = (e^{3i\theta}, e^{i\phi})$, we can take $N$ to be the same set $\{(z, w) \in \mathcal{C}^2 \ : \ |z| = |w| = 1\}$ but with a different conformal structure to make $f$ holomorphic. Here $\theta + i\phi$ is a local holomorphic coordinate for $M$ while $\frac{1}{3}\theta + i\phi$ is a local holomorphic coordinate for $N$. As seen in Exercise 4 from Assignment 5, there is no holomorphic map of degree 3 form $M$ to *itself.*

(d) $M$ is a torus and $N$ is a compact surface of genus 2. The map doesn't exist, because the Riemann-Hurwitz would imply that $1 = n(2-1) + 1 + \frac{B}{2}$ with $n \geq 1$, contradicting that $B$ is nonnegative.

(e) $M$ is a compact surface of genus 2 and $N$ is a torus. The map does exist. Riemann-Hurwitz only implies it must have total branching order 2. To see this map topologically, consider a usual picture of a genus 2 surface centered in $\mathbf{R}^3$ with symmetric handles on the left and right. Cut the surface by the XZ plane and spin the right half $180^o$ around the Z axis and reattach to obtain a torus. One can make the map holomorphic by lifting the conformal structure of the torus. The two branch points are on the Z-axis.

(f) $M$ is a compact surface of genus 3 and $N$ is a compact surface of genus 2. The map does exist. One can obtain it topologically by starting with a dgree 2 map of the torus to itself. Remove a small disk from the range as well as the two corresponding preimage disks from the domain. Then take two copies of the resulting map and sew together along corresponding circles in the domain and range. Again such a map (which now has no branch points) can be made holomorphic by choosing suitable conformal structures on the domain and range.

**3.** Suppose $f : M \to \mathbf{R}$ is a smooth function on a Riemann surface $M$. Show that the *energy density* 2 form

$$\Omega = \big[(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2\big]\, dx\, dy$$

is well-defined, independent of the local holomorphic coordinate $z = x + iy$. If $z = \phi(\tilde{z})$ where $\phi$ is biholomorphic, $z = x + iy$, and $\tilde{z} = \tilde{x} + i\tilde{y}$, then taking partial derivatives gives the matrix equation

$$\begin{pmatrix} \frac{\partial f}{\partial \tilde{x}} \\ \frac{\partial f}{\partial \tilde{y}} \end{pmatrix} = (D\phi) \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \,,$$

Hence,

$$(\frac{\partial f}{\partial \tilde{x}})^2 + (\frac{\partial f}{\partial \tilde{y}})^2 \;=\; |D\phi|^2 (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 \,.$$

Also

$$dx\, dy = \big(\det D\phi\big)\, \tilde{dx}\, \tilde{dy} \,.$$

Since $f$ is holomorphic,

$$|D\phi|^2 \;=\; |\phi'|^2 \;=\; \det D\phi \,,$$

hence

$$\big[(\frac{\partial f}{\partial \tilde{x}})^2 + (\frac{\partial f}{\partial \tilde{y}})^2\big]\, d\tilde{x}\, d\tilde{y} \;=\; \big[(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2\big]\, dx\, dy \,.$$

**4.** Find the Green's function $g(z, 1, 0, i)$ on $\hat{\mathbf{C}}$.

$$\log \Big|\frac{z-1}{z}\Big| \;-\; \frac{1}{2} \log 2 \,.$$

**5.** (a) Suppose $h$ is a meromorphic function on a compact Riemann surface $M$ of genus $g$. What is the dimension of the space of meromorphic functions $f$ on $M$

which have the *same* principal part as $h$? One. In fact, $f - g$ is holomorphic on $M$ and hence constant.

(b) Suppose $\beta$ is a meromorphic one form on a compact Riemann surface $M$ of genus $g$. What is the dimension of the space of meromorphic one forms $\alpha$ on $M$ which have the *same* principal part as $\beta$? $g$ . In fact, $\beta - \alpha$ is a holomorphic 1 form on $M$. The dimension of the space of holomorphic 1 forms on $M$ is $g$.

**6.** (a) Suppose $f$ is holomorphic on $\{z \in \mathbf{C} : 0 < |z| < 1\}$ and $|z| f(z) \to 0$ as $z \to 0$. Show that $f$ extends to be holomorphic across 0. By Morera's theorem, one need only show that $\int_\Gamma f(z)dz = 0$ for any closed curve $\Gamma$ in the punctured disk. By the Cauchy integral formula, it suffices to show that

$$\lim_{\delta \downarrow 0} \int_{\partial \mathbf{B}_\delta(0)} f(z)dz \;=\; 0 \;.$$

But this is bounded in absolute value by $\lim_{\delta \to 0} 2\pi\delta \sup_{\partial \mathbf{B}_\delta(0)} |f| = 0$.

(b) Is it also true that a harmonic function $h$ on $\{z \in \mathbf{C} : 0 < |z| < 1\}$ with $\lim_{z \to 0} |z| h(z) = 0$ also has a harmonic extension across 0. No, $\log |z|$ is harmonic with $\lim_{r \downarrow 0} r \log r = 0$.

(c) Is it also true that a holomorphic one-form $\omega$ on $\{z \in \mathbf{C} : 0 < |z| < 1\}$ with $\lim_{z \to 0} |z| \omega(z) = 0$ also has a holomorphic extension across 0. Yes, the quotient (as in #3 of HW8) $f = \frac{\omega}{dz}$ is a holomorphic function as in (a) with a removable singularity.

**7.** Recall the $L^2$ inner product on 1 forms:

$$(\alpha, \beta)_{L^2} \;=\; \int_M \alpha \wedge *\bar{\beta} \;.$$

Show that a *harmonic* 1 form $\omega$ uniquely minimizes $(\alpha, \alpha)_{L^2}$ in its $H^1(M)$ cohomology class, i.e.,

$$(\omega, \omega)_{L^2} \;\leq\; (\omega + df, \omega + df)_{L^2} \text{ with equality iff } df = 0 \;.$$

Note that $(\gamma, \gamma)_{L^2} \geq 0$ with equality iff $\gamma = 0$. Thus

$$\begin{aligned}
(\omega + df, \omega + df)_{L^2} &= (\omega, \omega)_{L^2} + (df, df)_{L^2} + (\omega, df)_{L^2} + (df, \omega)_{L^2} \\
&= (\omega, \omega)_{L^2} + (df, df)_{L^2} - \int df \wedge *\bar{\omega} + \int (d * \omega) \wedge \bar{f} \\
&= (\omega, \omega)_{L^2} + (df, df)_{L^2} + 0 + 0 \;\geq (\omega, \omega)_{L^2}
\end{aligned}$$

with equality iff $df = 0$.

**8.** Let $T = \{e^{i\theta}, e^{i\phi}) \ : \theta \in \mathbf{R}, \ \phi \in \mathbf{R}\}$ be the standard torus, and

$$\mathcal{P} = \{\mathcal{C}^\infty \text{ functions } f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}, \ f(\theta + 2\pi, \phi) = f(\theta, \phi) = f(\theta, \phi + 2\pi)\} \ .$$

(a) Show how any smooth 0 form $\alpha$ on $T$ corresponds to a function in $\beta \in \mathcal{P}$. Letting $\Pi : \mathbf{C} \to T$, $\Pi(\theta + i\phi)$, we see that $\alpha$ is a 0 form on $T$, i.e. a smooth function, iff $\beta = \alpha \circ \Pi \in \mathcal{P}$.

(b) Show how any smooth 1 form $\omega$ on $T$ corresponds to $fd\theta + gd\phi$ for some functions $f\, g \in \mathcal{P}$. The projection $\Pi$ can be used to "pull-back" $\omega$ to a 1 form on $\mathbf{C}$, which must look like $fd\theta + gd\phi$ for some smooth functions $f, g$ on the plane, which are necessarily doubly-periodic.

(c) Show how any smooth 2 form $\Omega$ on $T$ corresponds to $hd\theta d\phi$ for some function $h \in \mathcal{P}$. This is similar to (b).

(d) $\operatorname{grad} \beta = 0$ iff $\beta \equiv$ constant iff $\alpha \equiv$ constant.

(e) For which $f$, $g$ is $\omega$ closed? Since $d(fd\theta + gd\phi) = (-\frac{\partial f}{\partial \phi} + \frac{\partial g}{\partial \theta})d\theta d\phi$, $\omega$ is closed iff $\frac{\partial f}{\partial \phi} = \frac{\partial g}{\partial \theta}$.

(f) For which $f$, $g$ is $\omega$ exact? A closed form is locally exact. It will be globally exact iff all its periods are 0. Here we can take $a$ to be the oriented closed curve $\{(e^{i\theta}, 1) \ : \ 0 \le \theta \le 2\pi\}$ and $b$ to be the oriented closed curve $\{(1, e^{i\phi}) \ : \ 0 \le \phi \le 2\pi\}$. Then the periods $A = \frac{1}{2\pi}\int_a \omega$ and $B = \frac{1}{2\pi}\int_b \omega$ are the constant terms in the double Fourier series expansions of $f$ and $g$, respectively. The form $\omega$ will be exact iff $\omega$ is closed and $A = 0 = B$.

(g) For which $h$ is $\Omega$ exact? If $\Omega = d\omega$, then

$$0 \ = \ \int \Omega \ = \ \int_{[0,2\pi]\times[0,2\pi]} h(\theta, \phi) d\theta d\phi \ .$$

Conversely, if this integral is zero, then, for some smooth function $f$ on $T$, $\Omega = \text{Laplacian}\,(f) = d * df$, which is exact.

(h) For which $f$, $g$ is $\omega$ harmonic? From the Weyl decomposition or the Fourier series representation we see the $\omega = fd\theta + gd\phi$ is harmonic iff $f$ and $g$ are constants.

**9.** (a) What is the genus of the Riemann surface $Q$ associated with the equation $w^2 = z^3$? Zero. This is a 2 sheeted branched cover of the Riemann sphere (with branch points at 0 and $\infty$) and hence a sphere. Since the intersection with $|w| = r$ has only one component for all $r$ large, only one pointis added to obtain $Q$.

(b) What is the genus of the Riemann surface $R$ associated with $w^2 = z(z-1)$? This is a 2 sheeted branched cover of the Riemann sphere (with branch points at

0 and 1 and hence a sphere. Since the intersection with $|w| = r$ has 2 components for all $r$ large, two points are added to obtain $R$.

(c) What is the genus of the Riemann surface $S$ of $w^2 = z(z-1)(z-2)$? This is a 2 sheeted branched cover of the Riemann sphere (with branch points at 0, 1, 2, and $\infty$). Now the over-under identifications produce a torus. Since the intersection with $|w| = r$ has only one component for all $r$ large, only one point is added to obtain $S$.

**10.** Suppose $M$ is a compact connected Riemann surface of genus $g$. Recall that for a divisor $\mathcal{D}$ on $M$, $A(\mathcal{D})$ (respectively, $B(\mathcal{D})$) is the dimension of the space of meromorphic functions (respectively, 1 forms) which are multiples of $\mathcal{D}$.

(a) Let $\mathcal{D}(f)$ be the divisor of a nonconstant meromorphic function $f$ on $M$. $\deg \mathcal{D}(f) = 0$ by HW7, #2. Suppose that $g$ is a non-zero meromorphic function that is a multiple of $\mathcal{D}(f)^{-1} = \mathcal{D}(f^{-1})$. Then $g$ has at least as many zeroes and as most as many poles as $f^{-1}$. But for both $g$ and $f^{-1}$, the number of zeroes equals the number of poles. Thus the zeroes and poles of $g$ must coincide with those of $f^{-1}$ and $\frac{g}{f^{-1}}$ is a nonzero constant, hence, $A(\mathcal{D}(f)^{-1}) = 1$. The Riemann-Roch theorem then implies that $B(\mathcal{D}(f)) = g$.

(b) Let $\mathcal{D}(\omega)$ be the divisor of a meromorphic 1 form $\omega$ on $M$. Suppose $f$ is a multiple of $\mathcal{D}(\omega)^{-1}$. Then the zeroes of $f$ dominate the poles of $\omega$ and $f\omega$ is a *holomorphic* one form. Conversely, for any holomorphic 1 form $\alpha$, $\frac{\alpha}{\omega}$ is a meromorphic function that is a multiple of $\mathcal{D}(\omega)^{-1}$. Since there are $g$ linearly independent holomorphic one forms, $A(\mathcal{D}(\omega)^{-1}) = g$. Also if $\beta$ is a meromorphic 1 form that is a multiple of $\mathcal{D}(\omega)$, then $\frac{\beta}{\omega}$ is a *holomorphic* function, hence constant. Thus, $B(\mathcal{D}(\omega)) = 1$. The relation $\deg \mathcal{D}(\omega)) = 2g - 2$ then follows from the Riemann-Roch theorem.

HAPPY HOLIDAYS!