Riemann Roch Theorem

Suppose M is a compact connected Riemann surface of genus g.

A divisor on M is an element of the free abelian group of points of M. The usual representation is multiplicative:

$$\mathcal{D} \ = \ q_1^{m_1} q_2^{m_2} \dots q_j^{m_j} \ .$$

The degree of $\mathcal{D}$ is $m_1 + \dots + m_j$. We may write $\mathcal{D} = \mathcal{E}^{-1}\mathcal{F}$ where $\mathcal{E}$ and $\mathcal{F}$ are integral divisors, that is, ones with only nonnegative multiplicities. Let

$$A(\mathcal{D}) = \dim\{\text{meromophic functions that are multiples of } \mathcal{D}\}$$

$$B(\mathcal{D}^{-1}) = \dim\{\text{meromophic 1 forms that are multiples of } \mathcal{D}^{-1}\} \ .$$

Riemann-Roch Theorem.

$$A(\mathcal{D}) \ = \ B(\mathcal{D}^{-1}) \ - \ \deg \mathcal{D} \ - \ g \ + \ 1 \ .$$

Proof : We first consider the most important case:

Case 1. $\deg \mathcal{F} = 0$. Here $\mathcal{D} = \mathcal{E}^{-1}$, and we need to show that

$$A(\mathcal{E}^{-1}) \ = \ B(\mathcal{E}) \ + \ \deg \mathcal{E} \ - \ g \ + \ 1 \ .$$

Suppose $\mathcal{E} \ = \ p_1^{n_1} \dots p_k^{n_k}$. We will use the notion of principal parts. For *locally* defined meromorphic functions, these are easy to count. One sees that the vector space

$$P(\mathcal{E}^{-1}) = \{\text{principal parts of functions mero. } near \text{ spt}\,\mathcal{E} \text{ that are multiples of } \mathcal{E}\}$$

has

$$\dim P(\mathcal{E}^{-1}) \ = \ \deg \mathcal{E} \ ,$$

in fact, in suitable local coordinates,

$$\{(z - p_i)^{-j} \ : \ i = 1, \dots, k, \ j = 1, \dots, n_k\} \text{ spans } P(\mathcal{E}^{-1}) \ .$$

We are interested in the subspace

$$P_0 = \{\text{principal parts of } global \text{ mero. functions that are multiples of } \mathcal{E}^{-1}\} \ ,$$

Here

$$A(\mathcal{E}^{-1}) \ = \ \dim P_0 \ + \ 1 \qquad (*)$$

because the difference of 2 meromorphic functions with the same principal part is holomorphic, hence, constant, and the constant can be arbitrary.

To compute $\dim P_0$, recall that the space $\mathcal{H} = \{\text{global holomorphic 1 forms}\}$ has complex dimension g. Rephrasing our theorem (Royden, p.315) on prescribing principal parts, we have

Theorem. *$F \in P(\mathcal{E}^{-1})$ belongs to P_0 if and only if*

$$\int_\Gamma F\omega \;=\; 0 \;\text{ for all }\; \omega \in \mathcal{H} \;,$$

where $, \;= \sum_{i=1}^k \partial \mathbf{B}_\delta(p_i)$ *for δ small.*

A key question is "How many such ω do we need to test F?" While g such ω, given by a basis for $\mathcal{H}$, would do, we can actually use fewer. Let

$$\mathcal{H}(\mathcal{E}) \;=\; \{\text{holomorphic 1 forms that are multiples of } \mathcal{E}\} \;,$$

hence, $\dim \mathcal{H}(\mathcal{E}) \;=\; B \;=\; B(\mathcal{E})$. For $\omega \in \mathcal{H}(\mathcal{E})$ and *any* $F \in P(\mathcal{E}^{-1})$, $F\omega$ is holomorphic, hence $\int_\Gamma F\omega = 0$ automatically. Thus the ω in $\mathcal{H}(\mathcal{E})$ are not useful for testing membership in P_0.

On the other hand, if $\omega \in \mathcal{H} \setminus \mathcal{H}(\mathcal{E})$, then, at some p_i, ω vanishes to an order $n < n_i$ and $(z - p_i)^{-n-1}$ defines a principal part $\tilde{F} \in P(\mathcal{E}^{-1})$ so that $\tilde{F}\omega$ has a single simple pole at p_i and $\int_\Gamma \tilde{F}\omega \neq 0$. Choosing now a basis $\alpha_1, \ldots, \alpha_{g-B}$ for a space complementary to $\mathcal{H}(\mathcal{E})$ in $\mathcal{H}$, we conclude that the mapping

$$\Phi \;:\; P(\mathcal{E}^{-1}) \;\to\; \mathbf{C}^{g-B} \;,\quad \Phi(F) \;=\; \Big(\int_\Gamma F\alpha_1, \ldots, \int_\Gamma F\alpha_{g-B}\Big) \;,$$

is surjective with $\ker \Phi = P_0$, hence

$$\dim P_0 \;=\; \big(\deg \mathcal{E}\big) \;-\; \big(g \;-\; B(\mathcal{E})\big) \;.$$

Combining this with (*) completes the proof of Case 1.

Case 2. $\deg \mathcal{F} > 0$. Here we modify the argument of Case 1. First we take

$$P_0 = \{\text{principal parts of } \textit{global} \text{ mero. functions that are multiples of } \mathcal{D} = \mathcal{E}^{-1}\mathcal{F}\} \;,$$

and note that now

$$A(\mathcal{D}) \;=\; \dim P_0 \qquad (**)$$

because no nonzero constant function is a multiple of $\mathcal{D}$. Also the above theorem on prescribing principle parts now has the form

Theorem. *$F \in P(\mathcal{E}^{-1})$ belongs to P_0 if and only if*

$$\int_\Gamma F\omega \;=\; 0$$

for all meromorphic ω which are multiples of $\mathcal{F}^{-1}$.

This new version follows from the old version by using a Green's function representation (Royden, p.317). Now we are interested in the subspace $Q(\mathcal{F}^{-1})$ of meromorphic 1 forms ω which are multiples of $\mathcal{F}^{-1}$. Since there are g linearly independent holomorphic 1 forms and the principal parts, which are multiples of $\mathcal{F}^{-1}$, may be prescribed arbitrarily subject only to the one constraint that the sum of the residues is zero, we see that

$$\dim Q(\mathcal{F}^{-1}) \;=\; g \;+\; \deg \mathcal{F} \;-\; 1.$$

As in Case 1, the forms $\omega \in Q(\mathcal{F}^{-1})$ which automatically give $\int_\Gamma F\omega = 0$ for *all* $F \in P(\mathcal{E}^{-1})$ are in $Q(\mathcal{F}^{-1}\mathcal{E}) = Q(\mathcal{D}^{-1})$, which has dimension $B = B(\mathcal{D}^{-1})$. As in Case 1, we conclude that

$$\begin{aligned}
\dim P_0 \;&=\; \deg \mathcal{E} \;-\; \left[\dim Q(\mathcal{F}^{-1}) \;-\; B(\mathcal{D}^{-1})\right] \\
&=\; \deg \mathcal{E} \;-\left[g \;+\; \deg \mathcal{F} \;-\; 1 \;-\; B(\mathcal{D}^{-1})\right] \\
&=\; B(\mathcal{D}^{-1}) \;-\; \deg \mathcal{D} \;-\; g \;+\; 1\;.
\end{aligned}$$

Combining this with (**) completes the proof.

Some Consequences. Note that for $\mathcal{E}$ being an integral divisor, one has that $0 \le B(\mathcal{E}) \le g$ (because all the forms are holomorphic) and $A(\mathcal{E}^{-1}) \ge 1$ because the lack of conditions on the zeroes allow for constant functions.

In case $g = 0$, $B(\mathcal{E}) = 0$ and

$$A(\mathcal{E}^{-1}) \;=\; 1 \;+\; \deg \mathcal{E}\;.$$

In case $g = 1$, $B(\mathcal{E}) = 0$ because any nonzero holomorphic 1 form does not vanish. (On the standard torus it has the form $\lambda(d\theta + id\phi)$ for some $0 \ne \lambda \in \mathbf{C}$.) So now

$$A(\mathcal{E}^{-1}) \;=\; \deg \mathcal{E}\;.$$

In particular, by choosing $\mathcal{E} = p$ (as in Assignment 11, #3) or $\mathcal{E} = p^2$, we find that, *for any two points p, q on a torus T, there is a unique meromorphic function f on T having an order 2 pole at p and having $f(q) = 1$.*