Math 501, Topics in Differential Geometry
The Geometry of Banach Spaces

Meets: Tuesday-Thursday, 1-2:20 PM, Hermann Brown Hall 423

Description: A *metric space* is simply a set X with a "distance" function $d : X \times X \to [0, \infty)$ such that

$$d(w,x) = 0 \ \Leftrightarrow w = x, \quad d(x,y) = d(y,x), \quad d(x,z) \le d(x,y) + d(y,z) \quad \text{for all } w,x,y,z \in X,$$

On a vector space V, a norm $\| \ \| : V \to [0, \infty)$ is defined by the conditions

$$\|v\| = 0 \Leftrightarrow v = 0, \quad \|-v\| = \|v\|, \quad \|v + w\| \le \|v\| + \|w\| \quad \text{for all } v, w \in V$$

and defines the metric $d_{\| \ \|}(v,w) = \|v - w\|$ on V. The metric space (X,d) is *complete* if every d Cauchy sequence in X d converges, and the normed vector space $(V, \| \ \|)$ is a *Banach space* if the corresponding metric space $(V, d_{\| \ \|})$ is complete.

When are metric or Banach spaces the *same*? There are several notions of equivalence. For metric spaces there are 3 common notions: (X,d) is *isometric* to (Y,e) if there is a bijection ϕ of X onto Y preserving distances. The two metric spaces are *bilipschitz equivalent* if one has the weaker estimate $c^{-1}d(x,\tilde{x}) \le e\big(\phi(x), \phi(\tilde{x})\big) \le cd(x,\tilde{x})$ for some fixed $c \ge 1$. A still weaker notion is where ϕ is merely a *homeomorphism* so that the ϕ and ϕ^{-1} images of balls are open sets. With Banach spaces, one insists that $\phi : V \to W$ be *linear* and there is the somewhat confusing terminology that V and W are called *isomorphic* (respectively, *isometrically isomorphic*) if there is bilipschitz (respectively, *isometric*) linear bijection between them. I believe it is still unknown whether two bilipschitz separable Banach spaces are isomorphic (i.e. bilipschitz by a linear map). The finite dimensional case is true but nontrivial, involving Rademacher's theorem on the differentiability a.e. of a lipschitz map.

The first part of this course will be a fairly quick treatment of basic functional analysis covering e.g. notions of dual space, Hilbert space, weak or weak* topologies, the theorems of Hahn-Banach, open mapping, closed graph, uniform boundedness, Banach-Alaoglu, as well as looking at important classical Banach spaces $\ell^1, \ell^2, \ell^p, \ell^\infty, c_0, L^p(X,\mu), C([0,1]), C(Cantor), W^{1,p}(\mathbf{R^n})$.

Banach space theory is sometimes useful for studying general metric spaces. For example, *any* metric space admits a distance-preserving embedding into the Banach space $\ell^\infty(X)$ of bounded maps of X to $\mathbf{R}$. If moreover X is separable then it isometrically embeds into $C[0,1])$.

Some other topics include Hilbert space bases, Schauder bases, examples, complemented subspaces, lipschitz retracts, basic sequences, special properties of ℓ^p, $C(K)$, and $L^1(\mu)$, the Dunford-Pettis property, examples. To partially make up for our misuse of "Differential Geometry" in the course title, we will also consider some elementary results from nonlinear functional analysis including examples of Banach manifolds. We will try to occasionally make some references to a few of the numerous connections and applications of functional analysis to other fields such as PDE, combinatorics, or numerical analysis.

Main Reference: F. Albiac and N. Kalton *Topics in Banach Space Theory* Graduate Texts in Math 233, Springer, 2006.

Prerequisites: Linear Algebra, Point-set Topology, Beginning Analysis

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