Here we recall from lecture the following key results:

I) Solution of the Dirichlet problem for a ball and smoothness of harmonic functions (Poisson Integral formula)

II) Precompactness of any uniformly bounded family of harmonic functions

III) Characterization of subharmonicity by local sub-meanvalue inequalities

IV) Maximum principle for subharmonic functions

Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$.

**Definition.** For $b \in \partial\Omega$, a function $Q_b \in \mathcal{C}(\bar{\Omega})$ is a barrier at $b$ if $Q_b$ is subharmonic on $\Omega$, $Q_b(b) = 0$, and $Q_b(y) < 0$ for all $y \in \partial\Omega \setminus \{b\}$.

**Theorem.** If $\Omega$ has a barrier at each of its boundary points, then, for any $g \in \mathcal{C}(\partial\Omega)$, there exists a unique $u \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ such that

$$
\Delta u = 0 \text{ on } \Omega,
$$

$$
u = g \text{ on } \partial\Omega.
$$

**Proof:** Uniqueness follows from the maximum principle.

Let $m = \inf g$, $M = \sup g$, and

$$
\sigma_g = \{ \text{subharmonic } v \in \mathcal{C}(\bar{\Omega}) : v \leq g \text{ on } \partial\Omega \}.
$$

Then $\sigma_g$, containing the constant function $m$, is nonempty and

$$
u(x) = \sup_{v \in \sigma_g} v(x) \leq M \text{ for } x \in \bar{\Omega}
$$

is well-defined. We show that this $u$ satisfies the Theorem in 8 steps:

**STEP 1.** $v, \bar{v} \in \sigma_g$ implies $w = \max\{v, \bar{v}\} \in \sigma_g$.

Clearly $w \in \mathcal{C}(\bar{\Omega})$ and $w \leq g$ on $\partial\Omega$. For $B_r(a) \subset \Omega$

$$
w(a) = \max\{v(a), \bar{v}(a)\} \leq \max\{M_v(a, r), M_{\bar{v}}(a, r)\} \leq M_w(a, r).
$$

So $w$ is also subharmonic.

**STEP 2.** For $v \in \sigma_g$ and $B_r(a) \subset \Omega$, $v \leq v_{a,r} \in \sigma_g$ where

$$
\Delta v_{a,r} = 0 \text{ on } B_r(a),
$$

$$
v_{a,r} = v \text{ on } \bar{\Omega} \setminus B_r(a).
$$
The function \( v_{a,r} \) is obtained from \( I \), which implies that \( v_{a,r} \in C(\overline{\Omega}) \). Also \( v \leq v_{a,r} \) by the maximum principle. To see that \( v_{a,r} \) is subharmonic it suffices by \( III \) to show that, for each \( x \in \Omega \),

\[
v_{a,r}(x) \leq M v_{a,r}(x, s) \text{ for all sufficiently small positive } s .
\]

In case \( x \in B_r(a) \), take \( s < r - |x - a| \) so that

\[
v_{a,r}(x) = M v_{a,r}(x, s) \text{ by harmonicity.}
\]

In case \( x \in \Omega \setminus B_r(a) \), take \( s < \text{dist}(x, \partial \Omega) \) so that

\[
v_{a,r}(x) = v(x) \leq M v(x, s) \leq M v_{a,r}(x, s) .
\]

**STEP 3.** For any \( \overline{B_r(a)} \subset \Omega \) and countable \( X \subset B_r(a) \) there is a harmonic \( h \) on \( B_r(a) \) so that \( u(x) = h(x) \) for all \( x \in X \).

Fix \( s \) with \( r < s < \text{dist}(a, \partial \Omega) \) and write \( X = \{x_1, x_2, x_3, \ldots\} \). Choose \( v_i^j \in \sigma_g \) so that \( v_i^j(x_i) \uparrow u(x_i) \) as \( j \to \infty \). Then

\[
v^j \equiv \max\{m, v_1^j, v_2^j, \ldots, v_j^j\} \in \sigma_g \text{ by Step 1} ,
\]

\[
u^j \equiv v_{a,s}^j \in \sigma_g \text{ by Step 2} .
\]

Since \( m \leq u_{a,s}^j \leq M \), a subsequence \( v_{a,s}^{j_i} \) converges, by \( II \), uniformly on \( \overline{B_r(a)} \), to a harmonic \( h \). Also, since, for \( j \geq i \),

\[
v_i^j(x_i) \leq v^j(x_i) \leq v_{a,s}^j(x_i) \leq u(x_i) ,
\]

\[
h(x_i) = \lim_{j \to \infty} v_{a,s}^{j_i}(x_i) = \lim_{j \to \infty} v_{a,s}^j(x_i) = u(x_i)
\]

for all \( i \).

**STEP 4.** \( u \in C(\Omega) \). Suppose \( a \in \Omega \). For any positive \( r < \text{dist}(a, \partial \Omega) \) and any convergent sequence \( x_i \to a \) in \( B_r(a) \), we may apply Step 3 with \( X = \{a, x_1, x_2, \ldots\} \) to see that

\[
u(a) = h(a) = \lim_{i \to \infty} h(x_i) = \lim_{i \to \infty} u(x_i) .
\]

Thus \( u \) is continuous at \( a \).

**STEP 5.** \( u \) is harmonic on \( \Omega \). For any \( \overline{B_r(a)} \subset \Omega \) we apply Step 3 this time with \( X \) being a countable dense subset of \( B_r(a) \) to find a harmonic function \( \tilde{h} \) on \( B_r(a) \) coinciding with \( u \) on \( X \). But by Step 4, \( u \) as well as \( \tilde{h} \) is continuous. So, on \( B_r(a) \), \( u = \tilde{h} \) is harmonic.
Now we turn to the boundary behavior of $u$.

**STEP 6.** For each $b \in \partial \Omega$, $\liminf_{x \to b} u(x) \geq g(b)$.

For positive $\epsilon$ and $K$, note that the function

$$v(x) = g(b) - \epsilon + KQ_b(x) \text{ for } x \in \Omega$$

is continuous and subharmonic. Choose $\delta = \delta(\epsilon) > 0$ so that $g(x) > g(b) - \epsilon$ whenever $x \in \partial \Omega \cap B_\delta(b)$. Thus

$$v(x) \leq g(x) \text{ for } x \in \partial \Omega \cap B_\delta(b).$$

Since $Q_b(x)$ has a strictly negative upper bound on $\partial \Omega \setminus B_\delta(b)$, we can choose $K = K(\epsilon)$ large enough so that

$$v(x) \leq g(x) \text{ for } x \in \partial \Omega \setminus B_\delta(b).$$

Then we have $v \in \sigma_g$ so that $v \leq u$ and

$$g(b) - \epsilon = \lim_{x \to b} v(x) \leq \liminf_{x \to b} u(x).$$

**STEP 7.** For each $b \in \partial \Omega$, $\limsup_{x \to b} u(x) \leq g(b)$.

We turn things around by defining

$$\tilde{u}(x) = \sup_{-w \in \sigma_{-g}} -w(x) \text{ for } x \in \bar{\Omega},$$

and repeating Step 6 to conclude that $\liminf_{x \to b} \tilde{u}(x) \geq -g(b)$. For any $v \in \sigma_g$ and $-w \in \sigma_{-g}$, $v - w \leq 0$ on $\partial \Omega$ so that $v - w \leq 0$ on $\Omega$. Taking sup’s, we find that $u + \tilde{u} \leq 0$ or that

$$u \leq -\tilde{u}.$$

Thus

$$\limsup_{x \to b} u(x) \leq \limsup_{x \to b} -\tilde{u}(x) = -\liminf_{x \to b} \tilde{u}(x) \leq g(b).$$

**STEP 8.** $u \in C(\bar{\Omega}) \cap C^\infty(\Omega)$. Combine Steps 5, 6, 7.