Perron Method for the Dirichlet Problem

Here we recall from lecture the following key results:

I) Solution of the Dirichlet problem for a ball and smoothness of harmonic functions (Poisson Integral formula)

II) Precompactness of any uniformly bounded family of harmonic functions

III) Characterization of subharmonicity by local sub-meanvalue inequalities

IV) Maximum principle for subharmonic functions

Suppose that Ω is a bounded domain in $\mathbf{R}^n$.

Definition. For $b \in \partial\Omega$, a function $Q_b \in \mathcal{C}(\bar{\Omega})$ is a *barrier* at b if Q_b is subharmonic on Ω, $Q_b(b) = 0$, and $Q_b(y) < 0$ for all $y \in \partial\Omega \setminus \{b\}$.

Theorem. *If Ω has a barrier at each of its boundary points, then, for any $g \in \mathcal{C}(\partial\Omega)$, there exists a unique $u \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ such that*

$$\begin{aligned} \Delta u &= 0 \ \text{ on } \ \Omega \ , \\ u &= g \ \text{ on } \ \partial\Omega \ . \end{aligned}$$

Proof : Uniqueness follows from the maximum principle.

Let $m = \inf g$, $M = \sup g$, and

$$\sigma_g \ = \ \{\text{subharmonic } \ v \in \mathcal{C}(\bar{\Omega}) \ : \ v \leq g \text{ on } \ \partial\Omega\} \ .$$

Then σ_g, containing the constant function m, is nonempty and

$$u(x) \ = \ \sup_{v \in \sigma_g} v(x) \ \leq \ M \text{ for } \ x \in \bar{\Omega}$$

is well-defined. We show that this u satisfies the Theorem in 8 steps:

STEP 1. $v, \tilde{v} \in \sigma_g$ *implies* $w = \max\{v, \tilde{v}\} \in \sigma_g$.

Clearly $w \in \mathcal{C}(\bar{\Omega})$ and $w \leq g$ on $\partial\Omega$. For $\mathbf{B}_r(a) \subset \Omega$

$$w(a) = \max\{v(a), \tilde{v}(a)\} \leq \max\{M_v(a,r), M_{\tilde{v}}(a,r)\} \leq M_w(a,r) \ .$$

So w is also subharmonic.

STEP 2. *For $v \in \sigma_g$ and $\mathbf{B}_r(a) \subset \Omega$, $v \leq v_{a,r} \in \sigma_g$ where*

$$\begin{aligned} \Delta v_{a,r} &= 0 \ \text{ on } \ \mathbf{B}_r(a) \ , \\ v_{a,r} &= v \ \text{ on } \ \bar{\Omega} \setminus \mathbf{B}_r(a). \end{aligned}$$

The function $v_{a,r}$ is obtained from I, which implies that $v_{a,r} \in \mathcal{C}(\bar{\Omega})$. Also $v \leq v_{a,r}$ by the maximum principle. To see that $v_{a,r}$ is subharmonic it suffices by III to show that, for each $x \in \Omega$,

$$v_{a,r}(x) \ \leq \ M_{v_{a,r}}(x,s) \text{ for all sufficiently small positive } s\ .$$

In case $x \in \mathbf{B}_r(a)$, take $s < r - |x-a|$ so that

$$v_{a,r}(x) \ = \ M_{v_{a,r}}(x,s) \text{ by harmonicity.}$$

In case $x \in \Omega \setminus \mathbf{B}_r(a)$, take $s < \operatorname{dist}(x, \partial\Omega)$ so that

$$v_{a,r}(x) \ = \ v(x) \ \leq M_v(x,s) \ \leq M_{v_{a,r}}(x,s)\ .$$

STEP 3. *For any $\overline{\mathbf{B}_r(a)} \subset \Omega$ and countable $X \subset \mathbf{B}_r(a)$ there is a harmonic h on $\mathbf{B}_r(a)$ so that u(x)=h(x) for all $x \in X$.*

Fix s with $r < s < \operatorname{dist}(a, \partial\Omega)$ and write $X = \{x_1, x_2, x_3, \ldots\}$. Choose $v_i^j \in \sigma_g$ so that $v_i^j(x_i) \uparrow u(x_i)$ as $j \to \infty$. Then

$$v^j \ \equiv \ \max\{m, v_1^j, v_2^j, \ldots, v_j^j\} \ \in \ \sigma_g \text{ by Step } 1\,,$$

$$u^j \ \equiv \ v_{a,s}^j \ \in \ \sigma_g \text{ by Step } 2\,.$$

Since $m \leq v_{a,s}^j \leq M$, a subsequence $v_{a,s}^{j'}$ converges, by II, uniformly on $\overline{\mathbf{B}_r(a)}$, to a harmonic h. Also, since, for $j \geq i$,

$$v_i^j(x_i) \ \leq \ v^j(x_i) \ \leq \ v_{a,s}^j(x_i) \ \leq \ u(x_i)\ ,$$

$$h(x_i) = \lim_{j\to\infty} v_{a,s}^{j'}(x_i) = \lim_{j\to\infty} v_i^{j'}(x_i) = u(x_i)$$

for all i.

STEP 4. $u \in \mathcal{C}(\Omega)$. Suppose $a \in \Omega$. For any positive $r < \operatorname{dist}(a, \partial\Omega)$ and any convergent sequence $x_i \to a$ in $\mathbf{B}_r(a)$, we may apply Step 3 with $X = \{a, x_1, x_2, \ldots\}$ to see that

$$u(a) = h(a) = \lim_{i\to\infty} h(x_i) = \lim_{i\to\infty} u(x_i)\ .$$

Thus u is continuous at a.

STEP 5. *u is harmonic on* Ω. For any $\overline{\mathbf{B}_r(a)} \subset \Omega$ we apply Step 3 this time with X being a countable *dense* subset of $\mathbf{B}_r(a)$ to find a harmonic function $\tilde{h}$ on $\mathbf{B}_r(a)$ coinciding with u on X. But by Step 4, u as well as $\tilde{h}$ is continuous. So, on $\mathbf{B}_r(a)$, $u = \tilde{h}$ is harmonic.

Now we turn to the boundary behavior of u.

STEP 6. *For each* $b \in \partial\Omega$, $\liminf_{x \to b} u(x) \geq g(b)$.

For positive ϵ and K, note that the function

$$v(x) \;=\; g(b) - \epsilon + K Q_b(x) \text{ for } x \in \bar{\Omega}$$

is continuous and subharmonic. Choose $\delta = \delta(\epsilon) > 0$ so that $g(x) > g(b) - \epsilon$ whenever $x \in \partial\Omega \cap \mathbf{B}_\delta(b)$. Thus

$$v(x) \;\leq\; g(x) \text{ for } x \in \partial\Omega \cap \mathbf{B}_\delta(b) \; .$$

Since $Q_b(x)$ has a strictly negative upper bound on $\partial\Omega \setminus \mathbf{B}_\delta(b)$, we can choose $K = K(\epsilon)$ large enough so that

$$v(x) \;\leq\; g(x) \text{ for } x \in \partial\Omega \setminus \mathbf{B}_\delta(b) \; .$$

Then we have $v \in \sigma_g$ so that $v \leq u$ and

$$g(b) - \epsilon \;=\; \lim_{x \to b} v(x) \;\leq\; \liminf_{x \to b} u(x) \; .$$

STEP 7. *For each* $b \in \partial\Omega$, $\limsup_{x \to b} u(x) \leq g(b)$.

We turn things around by defining

$$\tilde{u}(x) \;=\; \sup_{-w \in \sigma_{-g}} -w(x) \text{ for } x \in \bar{\Omega} \; ,$$

and repeating Step 6 to conclude that $\liminf_{x \to b} \tilde{u}(x) \geq -g(b)$. For any $v \in \sigma_g$ and $-w \in \sigma_{-g}$, $v - w \leq 0$ on $\partial\Omega$ so that $v - w \leq 0$ on Ω. Taking sup's, we find that $u + \tilde{u} \leq 0$ or that

$$u \;\leq -\tilde{u}.$$

Thus

$$\limsup_{x \to b} u(x) \leq \limsup_{x \to b} -\tilde{u}(x) = -\liminf_{x \to b} \tilde{u}(x) \leq g(b) \; .$$

STEP 8. $u \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$. Combine Steps 5, 6, 7.

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