BV Compactness for Maps to a Metric Space

Suppose $-\infty \le a < b \le \infty$, E is a metric space, e_0 is any fixed point of E, and $f : (a,b) \to E$ is Lebesgue measurable.

Def. We say f belongs to $L^1\big((a,b)^n, E\big)$ if $\int_{(a,b)^n} \operatorname{dist}_E(f(x), e_0)\,dx < \infty$.

Lemma 1. *If $f_j,\ f : (a,b)^n \to E$ are $\mathcal{L}^n$ measurable and*

$$\Lambda_j \equiv \int_{(a,b)^n} \operatorname{dist}_E(f_j(x), f(x))\,dx \to 0 \text{ as } j \to \infty\ ,$$

then a subsequence $f_{j'}$ converges pointwise a.e. to f.

Proof : Choose a subsequence $f_{j'}$ so that $\sum_{j=1}^\infty \Lambda_j < \infty$. Then, since

$$\int_{(a,b)^n} \sum_{j=1}^\infty \operatorname{dist}_E(f_j(x), f(x))\,dx \ < \ \infty\ ,$$

$\sum_{j=1}^\infty \operatorname{dist}_E(f_j(x), f(x)) < \infty$ for a.e. $x \in (a,b)^n$ and $f_{j'}(x) \to 0$ for all such x. ∎

For a measurable map $f : (a,b) \to E$, we define the *essential variation*

$$\operatorname{ess} V_a^b(f) = \sup\{\sum_{i=1}^m \operatorname{dist}_E\big(f(t_i), f(t_{i-1})\big)\ : a < t_0 < t_1 < \ldots < t_m < b,$$
$$t_i \text{ are Lebesgue pts of } f\}\ .$$

Suppose $f \in L^1\big((a,b), \mathbf{R}\big)$ and $\operatorname{ess} V_a^b(f) < \infty$. Then f equals a.e. the difference of the two monotone functions $\operatorname{ess} V_a^x(f) \ - \ \big[\operatorname{ess} V_a^b(f) \ - \ f(x)\big]$. It follows that the limit $\tilde f(x) = \lim_{r\downarrow 0} \frac{1}{|\mathbf{B}_r(x)|}\int_{\mathbf{B}_r(x)} f(y)\,dy$ exists at *all* points in (a,b) and is continuous except for an atmost countable set where the left and right limits still exist but are different. By Lebesgue's differentiation theorem, $f = \tilde f$ a.e., and so $\operatorname{ess} V_a^b(f)$ coincides with the classical variation of $\tilde f$:

$$V_a^b(\tilde f) = \sup\{\sum_{i=1}^m \operatorname{dist}_E\big(\tilde f(t_i), \tilde f(t_{i-1})\big)\ : a < t_0 < t_1 < \ldots < t_m < b\}\ .$$

For an open set $U \subset \mathbf{R}^n$, recall also the distribution definition:

Def. $f \in BV(U) \ \Leftrightarrow \ f \in L^1(U)$ and

$$\|Df\|(U) \equiv \sup\{\int f \operatorname{div} w\ :\ w \in \mathcal{C}_0^\infty(U, \mathbf{R}^n),\ |w| \le 1\} \ < \ \infty\ .$$

For $n = 1$ we have:

Theorem 1. *If $f \in L^1\big((a,b)\big)$, then $\|Df\|\big((a,b)\big) \ = \ \operatorname{ess} V_a^b(f)$.*

The proof given in class, used smoothing for both implications and followed Evans-Gariepy, §5.10.1. ∎

Lemma 2. *Suppose* $0 < M < \infty$ *and* $f_j : [a,b] \to [-M,M]$ *are monotone increasing. Then a subsequence* $f_{j'}$ *converges pointwise off a countable set. Moreover,* $\|f_{j'} - f\|_{L^p} \to 0$ *for any* $p \in [1,\infty)$.

Proof : Suppose $\mathbf{Q} \cap \big[a,b\big] = \{a_1, a_2, \ldots\}$. A subsequence $f_{\alpha_1(1)}(a_1), f_{\alpha_1(2)}(a_1), \ldots$ of the bounded sequence of numbers $f_1(a_1), f_2(a_1), \ldots$ converges to a number $f(a_1)$. Inductively, choose a subsequence $f_{\alpha_j(1)}(a_j), f_{\alpha_j(2)}(a_j), \ldots$ of the sequence $f_{\alpha_{j-1}(1)}(a_j), f_{\alpha_{j-1}(2)}(a_j), \ldots$ convergent to a number $f(a_j)$.

Let $j' = \alpha_j(j)$ and $f(x) = \sup_{a_i < x} f(a_i) = \lim_{\epsilon \downarrow 0} \sup_{x-\epsilon < a_i < x} f(a_i)$. Then f is monotone increasing and the set Z of discontinuities of f is at most countable. To see that $\lim_{j\to\infty} f_{j'}(x) = f(x)$ for any $x \in (a,b) \setminus Z$, we choose, for $\epsilon > 0$, numbers $a_i < x < a_{\tilde{i}}$ so that $f(a_{\tilde{i}}) - \epsilon < f(x) < f(a_i) + \epsilon$, and then J so that

$$|f_{j'}(a_i) - f(a_i)| < \epsilon \text{ and } |f_{j'}(a_{\tilde{i}}) - f(a_{\tilde{i}})| < \epsilon$$

for $j \geq J$. For such j it follows that

$$f(x) - 2\epsilon < f(a_i) - \epsilon < f_{j'}(a_i) < f_{j'}(x) < f_{j'}(a_{\tilde{i}}) < f(a_{\tilde{i}}) + \epsilon < f(x) + 2\epsilon\ .$$

Thus $|f_{j'}(x) - f(x)| < 2\epsilon$.

To verify the second conclusion note that $|f_{j'} - f|^p \leq 2^p M^p$ and apply the Lebesgue dominated convergence theorem. ∎

Corollary 1. *Any sequence of functions* $f_j \in L^1\big((a,b)\big)$ *with*

$$M \equiv \sup_j \int_a^b |f_j|\, dx \ + \ \|Df_j\|\big((a,b)\big) \ < \ \infty$$

contains a subsequence $f_{j'}$ *convergent pointwise a.e. and in* L^1 *to a function* f *with* $\int_a^b |f|\, dx + \|Df\|\big((a,b)\big) \leq M$.

Proof : For the normalized functions $\tilde{f}_j$ we have the sup bound

$$\begin{aligned} |\tilde{f}_j| \ &\leq \ |b-a| V_a^b \tilde{f}_j \ + \ \inf |\tilde{f}_j| \\ &\leq \ |b-a| \|Df_j\|\big((a,b)\big) \ + \ \frac{1}{|b-a|} \int_a^b |f_j| \ \leq \ M(|b-a| + |b-a|^{-1})\ . \end{aligned}$$

We may as above then write $\tilde{f}_j$ as the difference $g_j - h_j$ of two uniformly bounded monotone increasing functions. Applying Lemma 2 to g_j and h_j gives the Corollary. ∎

A metric space E is called *weakly separable* if there is a sequence of functions $\phi_i : E \to \mathbf{R}$ with $\mathrm{Lip}\,(\phi_i) \le 1$ so that

$$\mathrm{dist}_E(x,y) \;=\; \inf_i |\phi_i(x) - \phi_i(y)| \text{ for all } \; x, y \in E \qquad (*).$$

A separable metric space E is weakly separable as one sees by taking $\phi_i(x) = \mathrm{dist}_E(x, e_i)$ for some countable dense subset $\{e_i\}$ of E. The dual space B^* of a separable Banach space is weakly separable as one sees by taking ϕ to be evaluation at b_i for some countable dense subset $\{b_i\}$ of the unit sphere of B. In particular the well-known space ℓ^∞ of bounded sequences (which is the dual space of the separable space ℓ^1) is weakly separable. One can also verify this directly by taking $\phi\big((a_1, a_2, \cdots)\big) = a_i$ for $(a_1, a_2, \cdots) \in \ell^\infty$.

In general, *a metric space E is weakly separable if and only if there is a distance preserving embedding of E into ℓ^∞.* With ϕ_i as in (*), one such embedding is

$$\iota(x) \;=\; \big(\phi_1(x) - \phi_1(e_0), \phi_2(x) - \phi_2(e_0), \ldots\big)$$

where e_0 is any given point of E.

We will say a map $F : E \to [0, \infty)$ is *lower proper*, if F is lower semi-continuous and $F^{-1}[0, R]$ is compact for every $R > 0$.

Corollary 2. (BV compactness for $n = 1$) *Suppose E is a weakly separable metric space, $F : E \to [0, \infty)$ is lower proper, and $0 < a < b < \infty$. Any sequence of functions $f_j \in L^1\big((a, b), E\big)$ with*

$$M \equiv \sup_j \int_a^b F\big(f_j(x)\big)\, dx \;+\; \mathrm{ess}\, V_a^b(f_j) \;<\; \infty$$

contains a subsequence $f_{j'}$ convergent pointwise a.e. and in L^1 to a function f with $\int_a^b F\big(f(x)\big)\, dx + \mathrm{ess}\, V_a^b(f)\big) \le M$.

Proof : We may assume E is a subset of ℓ^∞ and write $f_j = (f_j^1, f_j^2, \ldots)$. Passing to a subsequence, we find a Lebesgue point $c \in (a, b)$ so that $\sup_j |F\big(f_j(c)\big)| < \infty$. The lower properness implies that $\sup_j \|f_j(c)\|_{\ell^\infty} < \infty$. Combining this with the essential variation bound as in the proof of Corollary 1, we find that

$$\sup_j \|f_j\|_{L^\infty((a,b),\ell^\infty)} \;<\; \infty \;.$$

We may now apply Corollary 1 first to the sequence $(f_1^1, f_2^1, \ldots)$ to obtain a subsequence $f^1_{\alpha_1(j)}$ that is L^1 and pointwise a.e. convergent to some f^1, then inductively to the sequence $f^k_{\alpha_{k-1}(j)}$ to obtain a subsequence $f^k_{\alpha_k(j)}$ convergent to some f^k. We conclude that, for the diagonal sequence $f_{j'} = f_{\alpha_j(j)}$, each $f^k_{j'}$ is L^1 and pointwise a.e. convergent to f^k as $j \to \infty$. But, for the convergence of the functions $\|f_{j'}(x) - f(x)\|_{\ell^\infty}$, we still need to show that the rates of the convergences of $f^k_{j'}(x)$ to $f^k(x)$ are uniform independent of k and that the limit $f(x) \in E$ for almost every $x \in (a,b)$. For this purpose we will show that, for a.e. $x \in (a,b)$, any subsequence j'' of j' contains a subsequence j''' so that $f_{j'''}(x) \to f(x)$. First Fatou's Lemma gives that, for a.e. $x \in (a,b)$, $\liminf_{j''\to\infty} \operatorname{dist}_E(f_{j''}(x), e_0) < \infty$. So there is, by the bounded compactness assumption, a subsequence j''' of j'' (depending on x) and a limit point $e \in E$ so that $\|f_{j'''}(x) - e\|_{\ell^\infty} \to 0$ as $j \to \infty$. But the previous convergences of the components $f^k_{j'}$ show that $e = (e^1, e^2, \ldots) = (f^1(x), f^2(x), \ldots) = f(x)$. Thus we obtain the desired pointwise a.e. convergence. The L^1 convergence then follows by Lebesgue dominated convergence as in the proof of Corollary 1, and one readily checks that

$$\begin{aligned}\int_a^b F\big(f(x)\big)\,dx + \operatorname{ess} V_a^b(f) &\le \int_a^b \liminf_{j\to\infty} F\big(f_j(x)\big)\,dx \;+\; \operatorname{ess} V_a^b(f)\\ &\le \liminf_{j\to\infty} \int_a^b F\big(f_j(x)\big)\,dx + \operatorname{ess} V_a^b(f) \;\le\; M\ .\end{aligned}$$

■

Now we turn again to functions of n variables. For $x \in \mathbf{R}^n$, $k \in \{1, \ldots, n\}$, and $f : (a,b)^n \to E$, we define

$$\hat{x}_k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n),\;\; f_{(k)}(\hat{x}_k, x_k) = f(x)\ .$$

Then we have:

Lemma 3. *For $f \in L^1\big((a,b)^n, E\big)$ with E weakly separable, the mapping*

$$y \mapsto \operatorname{ess} V_a^b f_{(k)}(y, \cdot)$$

is $\mathcal{L}^{n-1}$ measurable.

Theorem 2. *Suppose $f \in L^1\big((a,b)^n\big)$. Then*

$$\|Df\|\big((a,b)^n\big) \;\le\; \sum_{k=1}^n \int_{(a,b)^{n-1}} \operatorname{ess} V_a^b f_{(k)}(y,\cdot)\,dy \;\le\; n\|Df\|\big((a,b)^n\big)\ .$$

The proofs given in class followed Evans-Gariepy, §5.10.2. ■

Based on the above, it is reasonable to say that a function $f \in L^1\big((a,b)^n, E\big)$ belongs to $BV\big((a,b)^n, E\big)$ if the *variation on lines*

$$VL(f) \;\equiv\; \sum_{k=1}^n \int_{(a,b)^{n-1}} \operatorname{ess} V_a^b f_{(k)}(y,\cdot)\,dy \;<\; \infty$$

and prove the following:

Theorem 3. (BV compactness) *Suppose E is a weakly separable metric space, $F : E \to [0,\infty)$ is lower proper, and $0 < a < b < \infty$. Any sequence of functions $f_j \in L^1\big((a,b)^n, E\big)$ with*

$$M \equiv \sup_j \int_{(a,b)^n} F\big((f_j(x)\big)\, dx \;+\; VL(f_j) \quad < \quad \infty$$

contains a subsequence $f_{j'}$ convergent pointwise a.e. and in L^1 to a function f with

$$\int_{(a,b)^n} F\big((f(x)\big)\, dx \;+\; VL(f) \quad \leq \; M \; .$$

Proof : We argue by induction on n. The case $n = 1$ follows from Corollary 2. Assuming the theorem true for dimensions less than n, we will use the weakly separable metric space

$$\tilde{E} \equiv \{h \in L^1\big((a,b)^{n-1}, E\big) \; : \; \int_{(a,b)^{n-1}} \tilde{F}\big((h\big)\, dy \;+\; VL(h) \; < \; \infty\}$$

where the functional $\tilde{F}(h) = \int_{(a,b)^{n-1}} F\big((h(x)\big)\, dx$ is, by induction, lower proper on $\tilde{E}$.

For each $j = 1, 2, \cdots$, and each $k \in \{1, \ldots, n\}$, consider the function

$$f_{j(k)} \;=\; (f_j)_{(k)} \;:\; (a,b)^{n-1} \times (a,b) \;\to\; E \; , \quad f_{j(k)}(\tilde{x}_k, x_k) = f_j(x) \; .$$

Fubini's theorem implies that, for a.e. $t \in \mathbf{R}$, each function $f_{j(k)}(\cdot, t) \in \tilde{E}$ and that the map $t \mapsto f_{j(k)}(\cdot, t)$ belongs to $L^1\big((a,b), \tilde{E}\big)$ with $\int_a^b \tilde{F}\big(f_{j(k)}(\cdot, t)\big)\, dt$ uniformly bounded by M. Also for $a \leq s < t \leq b$,

$$\operatorname{dist}_{\tilde{E}}\big(f_{j(k)}(\cdot, s), f_{j(k)}(\cdot, t)\big) \;=\; \int_{(a,b)^{n-1}} \operatorname{dist}_E\big(f_{j(k)}(y, s), f_{j(k)}(y, t)\big)\, dy$$

so that

$$\operatorname{ess} V_a^b f_{j(k)}(\cdot, \cdot) \;=\; \int_{(a,b)^{n-1}} \operatorname{ess} V_a^b f_{j(k)}(y, \cdot)\, dy \;\leq\; VL(f) \;\leq\; M \; .$$

The case $n = 1$ now gives L^1 convergence of a subsequence $f_{j'(k)}(\cdot, t)$ to a function $g_k \in BV\big((a,b), \tilde{E}\big)$. We obtain $g_1, \; g_2, \; \ldots \; , g_n$ by taking consecutive subsequences. The compatibility condition of the approximating functions

$$f_{j(k)}(x_k', x_k) \;=\; f_j(x_1, \ldots, x_n) \;=\; f_{j(l)}(x_l', x_l)$$

for $k, l \in \{1, \ldots, n\}$ along with Lemma 1 implies the compatibility of these limit functions

$$g_k(x_k', x_k) \;=\; g_l(x_l', x_l)$$

which implies, using Fubini's Theorem, the existence of a function well-defined by

$$f(x_1, \ldots, x_n) = g_k(x'_k, x_k)$$

for all $k = 1, \ldots, n$ and almost all $x \in (a,b)^n$. Also one has, by Fubini's theorem the L^1 convergence

$$\int_{(a,b)^n} \operatorname{dist}_E(f_{j'}, f)\, dx = \int_a^b \operatorname{dist}_{\tilde{E}}\big(f_{j'(k)}(\cdot, t), f_{(k)}(\cdot, t)\big)\, dt \;\to\; 0$$

as $j \to \infty$. By Lemma 1, a subsequence of the $f_{j'}$ also converges pointwise a.e. to f. Measurability is a consequence of the pointwise convergence. One readily verifies the estimate

$$\int_{(a,b)^n} F\big(f\big)\, dx + VL(f) \leq \; M$$

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