Compactness of Rectifiable Currents in a Metric Space

Suppose $u \in BV(\mathbf{R}^k)$ we define the *differential maximal function*

$$MDu(x) \ = \ sup_{r>0}\frac{1}{\omega_k r^k}\|Du\|\big(\mathbf{B}_r(x)\big) \ .$$

Lemma 1. *$\mathcal{L}^k(A_j) \le \frac{c(k)}{j}\|Du\|(\mathbf{R}^k)$ where $A_j \ = \ \{x \ : \ MDu(x) > j\}$. and $c(k)$ depends only on k.*

Proof : We may assume that u has compact support. For $x \in A_j$, there exists an $r(x) > 0$ so that $\|Du\|\big(\mathbf{B}_{r(x)}(x)\big) > j\omega_k r(x)^k$. The Besicovitch covering gives disjointed subfamilies of these balls $\mathcal{B}_1, \ldots, \mathcal{B}_{c(n)}$ whose unions altogether cover A_j. Thus

$$\begin{aligned}\mathcal{L}^k(A_j) \ &\le \sum_{i=1}^{c(k)} \sum_{B_{r_j}(x_j)\in\mathcal{B}_i} \omega_k r_j^k \\ &\le \sum_{i=1}^{c(k)} \frac{1}{j} \sum_{B_{r_j}(x_j)\in\mathcal{B}_i} \|Du\| B_{r_j}(x_j) \ \le \ \frac{c(k)}{j}\|Du\|(\mathbf{R}^k) \ .\end{aligned}$$

■

Lemma 2. *If x is a Lebesgue point of u, then*

$$\frac{1}{|\mathbf{B}_\rho|}\int_{\mathbf{B}_\rho(x)} \frac{|u(z)-u(x)|}{|z-x|}\,dz \ \le \ MDu(x) \ .$$

Proof : We may assume that $\rho = 1$, that $x = 0$, and , by smoothing, that $u \in \mathcal{C}^1$. Then since $u(z) - u(0) \ = \ \int_0^1 Du(tz)\cdot z\,dt$,

$$\begin{aligned}\int_{\mathbf{B}_1}\frac{|u(z)-u(0)|}{|z|}\,dz \ &\le \ \int_{\mathbf{B}_1}\int_0^1 |Du(tz)|\,dt\,dz \ = \ \int_0^1\int_{\mathbf{B}_1}|Du(tz)|\,dz\,dt \\ &= \ \int_0^1 t^{-k}\int_{\mathbf{B}_t}|Du(z)|\,dz\,dt \ \le \ \omega_k MDu(0) \ .\end{aligned}$$

■

Theorem 1. (BV→Lipschitz) *If $u \in BV(\mathbf{R}^k, E)$ with E weakly separable, then there is an $\mathcal{L}^k$ null set N so that for $x, y \in N$*

$$\text{dist}_E\big(u(x), u(y)\big) \ \le \ C(k)\big[MDu(x) + MDu(y)\big]|x-y| \ .$$

Proof : We may assume that $E \subset \ell^\infty$. Let N be the union of all the non-Lebesgue points of the components $u^1, u^2, \ldots$ of u. Thus $\mathcal{L}^k(N) \ = \ 0$. Let $\rho \ = \ |x-y|$. For $z \in A \equiv \mathbf{B}_\rho(x)\cap\mathbf{B}_\rho(y)$, we can estimate

$$\frac{|u^i(x)-u^i(y)|}{|x-y|} \ \le \ \frac{|u^i(x)-u^i(z)|}{|x-z|} \ + \ \frac{|u^i(z)-u^i(y)|}{|z-y|} \ .$$

Thus, by Lemma 2,

$$\begin{aligned}
&\frac{1}{|A|}\int_A \frac{|u^i(x)-u^i(y)|}{|x-y|} \\
&\le\; C\Big(\frac{1}{|\mathbf{B}_\rho|}\int_{\mathbf{B}_\rho(x)} \frac{|u^i(x)-u(z)|}{|x-z|}\,dz \;+\; \frac{1}{|\mathbf{B}_\rho|}\int_{\mathbf{B}_\rho(y)} \frac{|u^i(y)-u^i(x)|}{|y-x|}\,dy\Big) \\
&\le\; C\big(MDu(x) \;+\; MDu(y)\big)\ .
\end{aligned}$$

The Theorem follows by taking the supremum over $i = 1, 2, \ldots$. ■

Corollary 1. *For $\epsilon > 0$ there exists a $v \in \mathrm{Lip}\,(\mathbf{R}^k, \ell^\infty)$ so that $\mathcal{L}^k\{x\ : v(x) \neq u(x)\} < \epsilon$.*

Recall $\mathcal{R}_k(E)$ denotes the set of all integer-multiplicity k rectifiable currents in E equipped with the *flat metric*

$$\mathrm{dist}_{\,F}(T,\tilde{T}) \;=\; \inf\{\mathbf{M}(R)+\mathbf{M}(\partial S)\ :\ T-\tilde{T}=R+\partial S,\ R\in\mathcal{R}_k(E),\ S\in\mathcal{R}_{k+1}(E)\}\ ,$$

and that $\mathcal{R}_0$ is simply the set of finite sums of integral multiples of point masses in E.

Theorem 2. *If $\mathcal{S} \in BV(\mathbf{R}^k, \mathcal{R}_0)$, then, for some $\mathcal{L}^k$ null set N in $\mathbf{R}^k$, the set $\cup_{y\in\mathbf{R}^k\setminus N}\,\mathrm{spt}\,\mathcal{S}(y)$ is k rectifiable.*

Proof : By the finiteness of $\mathrm{spt}\,\mathcal{S}(y)$, the number

$$\delta(y) = 4 \min_{x\neq\tilde{x}\in\mathrm{spt}\,\mathcal{S}(y)} \mathrm{dist}_{\,E}(x,\tilde{x})$$

is positive. Let

$$Y_{i,j} \;=\; \{y\in\mathbf{R}^k\ :\ MD\mathcal{S}(y)\le i,\ \delta(y) > \frac{1}{j}\}$$

so that $N = \mathbf{R}^k \setminus \cup_{i,j=1}^\infty Y_{i,j} = 0$. If $y, \tilde{y} \in Y_{i,j}$ and $|y-\tilde{y}| < \frac{1}{2Cij}$, then

$$\mathrm{dist}\,\big(\mathcal{S}(y),\mathcal{S}(\tilde{y})\ <\ \frac{1}{2Cij}C(i+i)\ \le\ \frac{1}{j}\ .$$

For each $x \in \mathrm{spt}\,\mathcal{S}(y)$, there exists a unique point $\mathbf{B}_{1/j}(x)\tilde{x} \in \mathrm{spt}\,\mathcal{S}(\tilde{y})$ and vice versa. Moreover,

$$\mathrm{dist}_{\,E}(x,\tilde{x})\ \le\ \mathrm{dist}_{\,F}\big(\mathcal{S}(y),\mathcal{S}(\tilde{y})\big)\ .$$

So locally $\cup_{y\in Y_{i,j}}\mathrm{spt}\,\mathcal{S}(y)$ is a finite union of Lipschitz graphs. This implies that

$$\cup_{i,j=1}^\infty \cup_{y\in Y_{i,j}}\ \mathrm{spt}\,\mathcal{S}(y)$$

is k rectifiable. ■

Corollary 2. *If* $T \in \mathcal{N}_k(E)$ *and, for any Lipschitz maps* $\pi : E \to \mathbf{R}^k$, $< T, \pi, y > \in \mathcal{R}_0(E)$ *for a.e.* $y \in \mathbf{R}^k$*, then* $T \in \mathcal{R}_k(E)$.

Proof : Recall that $\mathcal{S} =< T, \pi, \cdot > \in BV\big(\mathbf{R}^k, \mathcal{R}_0(E)\big)$. Since $\|T \llcorner d\pi\| = \int_{\mathbf{R}^k} \|\mathcal{S}(y)\|\, dy$, each measure $\|T \llcorner d\pi\|$ is carried by the k rectifiable set $\cup_{y \in \mathbf{R}^k \setminus N} \operatorname{spt} \mathcal{S}(y)$. Moreover, $\|T\|$ is a supremum over the measures $\|T \llcorner d\pi\|$ corresponding to π with $\operatorname{Lip}(\pi^i) \le 1$. Since this supremum can be taken over a countable collection of such π, $\|T\|$ is a k rectifiable measure, which implies that T is a k rectifiable current. ■

The compactness of *rectifiable currents* will follow from the Closure Theorem 4 below and

Theorem 3.(Normal Current Compactness) *Suppose* E *is a compact metric space and* $\mathcal{T}$ *is a family of* k *dimensional normal currents in* E *with*

$$\Lambda \;=\; \sup_{T \in \mathcal{T}} \mathbf{M}(T) + \mathbf{M}(\partial T) \;<\; \infty \; .$$

Then $\mathcal{T}$ *contains a convergent sequence* $T_i \to T$*, and* $\mathbf{M}(T) + \mathbf{M}(\partial T) \le \Lambda$.

Proof : As shown in class, we may use the diagonal trick to find a sequence T_i so that $T_i(f\, d\pi)$ converges for all $f\, d\pi$ belonging to a countable dense subset of $\mathcal{D}^k(E)$. This defines $T(f\, d\pi)$ for such $f\, d\pi$. We then use an equicontinuity estimate

$$|S(f d\pi) - S(\tilde{f} d\tilde{\pi})| \le \; C(\Lambda) \sup_x |f(x) - \tilde{f}(x)| \; + C(\Lambda, \operatorname{Lip}\pi, \operatorname{Lip}\tilde{\pi}) \sup_x |\pi(x) - \tilde{\pi}(x)| \; ,$$

valid for all $S \in \mathcal{T}$, to extend the convergence to all of $\mathcal{D}^k(E)$. ■

Lemma 3. *If* $\pi : E \to \mathbf{R}^m$ *is Lipschitz with* $m \le k$*, then, for almost every* $y \in \mathbf{R}^m$,

$$\sup_i \mathbf{M} < T_{i'}, \pi, y > + \mathbf{M}\partial < T_{i'}, \pi, y > \;<\; \infty$$

and $< T_{i'}, \pi, y > \to < T, \pi, y >$ *for some subsequence* i' *(depending on* y*).*

Proof : Fatou's Lemma and the integral estimate

$$\int \big(\mathbf{M} < T_{i'}, \pi, y > + \mathbf{M}\partial < T_{i'}, \pi, y > \big) dy \;\le\; C(\pi)\big(\mathbf{M}(T_{i'}) + \mathbf{M}(\partial T_{i'})\big)$$

guarantees that we can, for almost any y pass to a subsequence to have

$$\sup_i \mathbf{M} < T_{i'}, \pi, y > + \mathbf{M}\partial < T_{i'}, \pi, y > \;<\; \infty \; .$$

To prove convergence at almost all slices, we argue by induction on m. For $m = 1$ we have, for all but countably many t, that

$$\big(\|T_i\| \;+\; \|T\| \;+\; \|\partial T_i\| \;+\; \|\partial T\|\big)(\pi^{-1}\{t\}) \;=\; 0 \; .$$

So

$$\begin{aligned} < T_i, \pi, t > \;&=\; (\partial T_i) \llcorner \{\pi < t\} \;-\; \partial(T_i \llcorner \{\pi < t\}) \\ &\to\; (\partial T) \llcorner \{\pi < t\} \;-\; \partial(T \llcorner \{\pi < t\}) \;= < T, \pi, t > \; . \end{aligned}$$

For $m > 1$ we repeat the argument. ■

Theorem 4. (Rectifiability Closure Theorem) *Suppose T_i and T are as in Theorem 3. If $T_i \in \mathcal{R}_k(E)$, then $T \in \mathcal{R}_k(E)$.*

Proof : The case $k = 0$ is elementary. Here $T_i = \sum_{j=1}^{k_i} n_{i,j}[a_{i,j}]$ where $a_{i,j}$ are points in K and $n_{i,j}$ are integers with $\sum_{j=1}^{k_i} |n_{i,j}| = \mathbf{M}(T_i) \leq \Lambda$. One sees that $\operatorname{spt} T$ is contained in the Hausdorff limit of a subsequence of the sets $\{a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}\}$ and so is a finite subset of K of at most Λ points. The convergence of T_i to T guarantees that, for each $a \in \operatorname{spt} T$, for

$$\delta < \frac{1}{2} \inf_{a \neq \tilde{a} \in \operatorname{spt} T} \operatorname{dist}_E(a, \tilde{a}) \ ,$$

and for i sufficiently large, the integer $n_a = T_i(\chi_{\mathbf{B}_\delta(a)})$ is independent of i. Thus, $T = \sum_{a \in \operatorname{spt} T} n_a[a] \in \mathcal{R}_0$.

For $k > 0$ and any Lipschitz $\pi : E \to \mathbf{R}^k$, we apply Lemma 3 and the case $k = 0$ to conclude that, for almost all $y \in \mathbf{R}^k$,

$$< T, \pi, y > = \lim_{i \to \infty} < T_{i'}, \pi, y > \in \ \mathcal{R}_0 \ .$$

By Corollary 2, $T \in \mathcal{R}_k(E)$. ∎

Corollary 3. (A Plateau Problem in ℓ^2) *Suppose $B \in \mathcal{R}_{k-1}(\ell^2)$, $\partial B = 0$, and $K = \operatorname{spt} B$ is compact. Then there exists a mass-minimizer in the family*

$$\mathcal{T}_B \ = \ \{T \in \mathcal{R}_k(\ell^\infty) \ : \ \partial T \ = \ B\} \ .$$

Proof : One easily checks that the cone $h_\#\big([0,1] \times B\big) \in \mathcal{T}_B$ where $h(t,x) = tx$. Thus $\mathcal{T}_B \neq \emptyset$. Also

$$\hat{K} \ = \ \text{convex hull}\,(K) \ = \ \{(1-t)x + ty \ : \ 0 \leq t \leq 1, \ x, y \in K\}$$

is compact. Let $r : \ell^\infty \to \hat{K}$ be the nearest point retraction. One readily checks that $\operatorname{Lip} r \leq 1$. Assuming $T_i \in \mathcal{T}_B$ is a mass-minimizing sequence, we see that $r_\# T_i \in \mathcal{R}_k(\ell^2)$ and that

$$\partial r_\# T_i = r_\# \partial T_i = r_\# B = B \ ,$$

hence, $r_\# T_i \in \mathcal{T}_B$. Moreover, $r_\# T_i$ is again a mass-minimizing sequence because $\mathbf{M}(r_\# T_i) \leq \mathbf{M}(T_i)$. Applying Theorem 4 with T_i, E replaced by $r_\# T_i, \hat{K}$ we find the convergence of a subsequence $r_\# T_{i'} \to T \in \mathcal{R}_k(E)$. Since $\partial T = \lim_{i \to \infty} \partial r_\# T_{i'} = B$ and $\mathbf{M}(T) \leq \liminf_{i \to \infty} \mathbf{M}\big(r_\# T_{i'}\big)$, T is the desired mass-minimizer in $\mathcal{T}_B$. ∎