BV Compactness for Maps to a Metric Space

Suppose $-\infty \leq a < b \leq \infty$, $E$ is a metric space, $e_0$ is any fixed point of $E$, and $f : (a, b) \rightarrow E$ is Lebesgue measurable.

**Def.** We say $f$ belongs to $L^1((a, b)^n, E)$ if $\int_{(a,b)^n} \text{dist}_E(f(x), e_0) \, dx < \infty$.

**Lemma 1.** If $f_j, f : (a, b)^n \rightarrow E$ are $L^n$ measurable and $\Lambda_j \equiv \int_{(a,b)^n} \text{dist}_E(f_j(x), f(x)) \, dx \rightarrow 0$ as $j \rightarrow \infty$,

then a subsequence $f_j'$ converges pointwise a.e. to $f$.

**Proof:** Choose a subsequence $f_j'$ so that $\sum_{j=1}^{\infty} \Lambda_j < \infty$. Then, since $\int_{(a,b)^n} \sum_{j=1}^{\infty} \text{dist}_E(f_j(x), f(x)) \, dx < \infty$,

$\sum_{j=1}^{\infty} \text{dist}_E(f_j(x), f(x)) < \infty$ for a.e. $x \in (a, b)^n$ and $f_j'(x) \rightarrow 0$ for all such $x$. ■

For a measurable map $f : (a, b) \rightarrow E$, we define the essential variation

$$\text{ess} V_b^a(f) = \sup \{ \sum_{i=1}^{m} \text{dist}_E(f(t_i), f(t_{i-1})) : a < t_0 < t_1 < \ldots < t_m < b,\ t_i \text{ are Lebesgue pts of } f \} .$$

Suppose $f \in L^1((a, b), \mathbb{R})$ and $\text{ess} V_b^a(f) < \infty$. Then $f$ equals a.e. the difference of the two monotone functions $\text{ess} V_b^a(f) - [\text{ess} V_b^a(f) - f(x)]$. It follows that the limit $\tilde{f}(x) = \lim_{r \downarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) \, dy$ exists at all points in $(a, b)$ and is continuous except for an atmost countable set where the left and right limits still exist but are different. By Lebesgue’s differentiation theorem, $f = \tilde{f}$ a.e., and so $\text{ess} V_b^a(f)$ coincides with the classical variation of $\tilde{f}$:

$$V_b^a(\tilde{f}) = \sup \{ \sum_{i=1}^{m} \text{dist}_E(\tilde{f}(t_i), \tilde{f}(t_{i-1})) : a < t_0 < t_1 < \ldots < t_m < b \} .$$

For an open set $U \subset \mathbb{R}^n$, recall also the distribution definition:

**Def.** $f \in BV(U) \iff f \in L^1(U)$ and

$$\|Df\|(U) \equiv \sup \{ \int f \text{ div } w : w \in C_0^\infty(U, \mathbb{R}^n), \ |w| \leq 1 \} < \infty .$$

For $n = 1$ we have:

**Theorem 1.** If $f \in L^1((a, b))$, then $\|Df\|((a,b)) = \text{ess} V_b^a(f)$.

The proof given in class, used smoothing for both implications and followed Evans-Gariepy, §5.10.1. ■
Lemma 2. Suppose $0 < M < \infty$ and $f_j : [a, b] \to [-M, M]$ are monotone increasing. Then a subsequence $f_{j'}$ converges pointwise off a countable set. Moreover, $\|f_{j'} - f\|_{L^p} \to 0$ for any $p \in [1, \infty)$.

Proof: Suppose $Q \cap [a, b] = \{a_1, a_2, \ldots\}$. A subsequence $f_{\alpha_1(1)}(a_1), f_{\alpha_1(2)}(a_1), \ldots$ of the bounded sequence of numbers $f_1(a_1), f_2(a_1), \ldots$ converges to a number $f(a_1)$. Inductively, choose a subsequence $f_{\alpha_j(1)}(a_j), f_{\alpha_j(2)}(a_j), \ldots$ of the sequence $f_{\alpha_{j-1}(1)}(a_j), f_{\alpha_{j-1}(2)}(a_j), \ldots$ convergent to a number $f(a_j)$.

Let $j' = \alpha_j(j)$ and $f(x) = \sup_{a_i < x} f(a_i) = \lim_{\epsilon \downarrow 0} \sup_{x-\epsilon < a_i < x} f(a_i)$. Then $f$ is monotone increasing and the set $Z$ of discontinuities of $f$ is at most countable. To see that $\lim_{j \to \infty} f_{j'}(x) = f(x)$ for any $x \in (a, b) \setminus Z$, we choose, for $\epsilon > 0$, numbers $a_i < x < a_j$ so that $f(a_i) - \epsilon < f(x) < f(a_i) + \epsilon$, and then $J$ so that

$$|f_{j'}(a_i) - f(a_i)| < \epsilon \quad \text{and} \quad |f_{j'}(a_j) - f(a_j)| < \epsilon$$

for $j \geq J$. For such $j$ it follows that

$$f(x) - 2\epsilon < f(a_i) - \epsilon < f_{j'}(a_i) < f_{j'}(x) < f_{j'}(a_j) < f(a_j) + \epsilon < f(x) + 2\epsilon.$$ Thus $|f_{j'}(x) - f(x)| < 2\epsilon$.

To verify the second conclusion note that $|f_{j'} - f|^p \leq 2^p M^p$ and apply the Lebesgue dominated convergence theorem.

Corollary 1. Any sequence of functions $f_j \in L^1((a, b))$ with

$$M \equiv \sup_j \int_a^b |f_j| \, dx + \|Df_j\|((a, b)) < \infty$$

contains a subsequence $f_{j'}$ convergent pointwise a.e. and in $L^1$ to a function $f$ with $\int_a^b |f| \, dx + \|Df\|((a, b)) \leq M$.

Proof: For the normalized functions $\tilde{f}_j$ we have the bound

$$|\tilde{f}_j| \leq |b - a| V^b_a \tilde{f}_j + \inf |\tilde{f}_j| \
\leq |b - a| \|Df_j\|((a, b)) + \frac{1}{|b - a|} \int_a^b |f_j| \leq M(|b - a| + |b - a|^{-1}).$$

We may as above then write $\tilde{f}_j$ as the difference $g_j - h_j$ of two uniformly bounded monotone increasing functions. Applying Lemma 2 to $g_j$ and $h_j$ gives the Corollary.
Recall that a metric space $E$ is weakly separable if there is a sequence of functions $\phi_i : E \to \mathbb{R}$ with $\text{Lip}(\phi_i) \leq 1$ so that

$$\text{dist}_E(x, y) = \inf_i |\phi_i(x) - \phi_i(y)| \text{ for all } x, y \in E.$$ 

A separable metric space $E$ is weakly separable as one sees by taking $\phi_i(x) = \text{dist}_E(x, e_i)$ for some countable dense subset $\{e_i\}$ of $E$. Moreover, $E$ is weakly separable if and only if there is a distance preserving embedding of $E$ into 

$$\ell^\infty = \{(a_1, a_2, \ldots) : \sup_i |a_i| < \infty\}.$$ 

With $\phi_i$ as above, one such embedding is 

$$\iota(x) = (\phi_1(x) - \phi_1(e_0), \phi_2(x) - \phi_2(e_0), \ldots)$$

where $e_0$ is any given point of $E$.

We also say $E$ is boundedly compact if every closed ball $\overline{B}_R(e_0) = \{x : \text{dist}_E(x, e_0) \leq R\}$ is compact for $0 < R < \infty$. This implies that $E$ is locally compact and complete.

**Corollary 2.** (BV compactness for $n = 1$) Suppose $E$ is a boundedly compact weakly separable metric space, $e_0 \in E$, and $0 < a < b < \infty$. Any sequence of functions $f_j \in L^1((a, b), E)$ with 

$$M \equiv \sup_j \int_a^b \text{dist}_E(f_j(x), e_0) \, dx + \text{ess} \sup V^b_a(f_j) < \infty$$

contains a subsequence $f_{j'}$ convergent pointwise a.e. and in $L^1$ to a function $f$ with 

$$\int_a^b \text{dist}_E(f(x), e_0) \, dx + \text{ess} \sup V^b_a(f) \leq M.$$

**Proof:** We may assume $E$ is a boundedly compact subset of $\ell^\infty$ and write $f_j = (f^1_j, f^2_j, \ldots)$.

We apply Corollary 1 first to the sequence $(f^1_j, f^2_j, \ldots)$ to obtain a subsequence $f^1_{\alpha_{1}(j)}$ that is $L^1$ and pointwise a.e. convergent to some $f^1$, then inductively to the sequence $f^k_{\alpha_{k-1}(j)}$ to obtain a subsequence $f^k_{\alpha_k(j)}$ convergent to some $f^k$. We conclude that, for the diagonal sequence $f_{j'} = f_{\alpha_{k}(j)}$, each $f^k_{j'}$ is $L^1$ and pointwise a.e. convergent to $f^k$ as $j \to \infty$. But, for the convergence of the functions $\|f_{j'}(x) - f(x)\|_{\ell^\infty}$, we still need to show that the rates of the convergences of $f^k_{j'}(x)$ to $f^k(x)$ are uniform independent of $k$ and that the limit $f(x) \in E$ for almost every $x \in (a, b)$. For this purpose we will show that, for a.e. $x \in (a, b)$, any subsequence $j'''$ of $j'$ contains a subsequence $j'''$ so that $f_{j''''}(x) \to f(x)$. First Fatou’s Lemma gives that, for a.e. $x \in (a, b)$, 

$$\liminf_{j' \to \infty} \text{dist}_E(f_{j'''}(x), e_0) < \infty.$$

So there is, by the bounded compactness assumption, a subsequence $j'''$ of $j''$ (depending on $x$) and a limit point $e \in E$ so that $\|f_{j'''}(x) - e\|_{\ell^\infty} \to 0$ as $j \to \infty$. But the previous convergences of the components $f^k_{j'}$ show that $e = (e^1, e^2, \ldots) = (f^1(x), f^2(x), \ldots) = f(x).$
Thus we obtain the desired pointwise a.e. convergence. The $L^1$ convergence then follows by Lebesgue dominated convergence as in the proof of Corollary 1, and one readily checks that

$$\int_a^b \|f(x)\|_{L^\infty} \, dx + \text{ess} \, V_a^b(f) \leq M.$$ 

Now we turn again to functions of $n$ variables. For $x \in \mathbb{R}^n$, $k \in \{1, \ldots, n\}$, and $f : (a, b)^n \to E$, we define

$$\hat{x}_k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \quad f_{(k)}(\hat{x}_k, x_k) = f(x).$$

Then we have:

**Lemma 3.** For $f \in L^1((a, b)^n, E)$ with $E$ weakly separable, the mapping

$$y \mapsto \text{ess} \, V_a^b f_{(k)}(y, \cdot)$$

is $L^{n-1}$ measurable.

**Theorem 2.** Suppose $f \in L^1((a, b)^n)$. Then

$$\|Df\|((a, b)^n) \leq \sum_{k=1}^n \int_{(a, b)^n} \text{ess} \, V_a^b f_{(k)}(y, \cdot) \, dy \leq n \|Df\|((a, b)^n).$$

The proofs given in class followed Evans-Gariepy, §5.10.2.

Based on the above it is reasonable to say that a function $f \in L^1((a, b)^n, E)$ belongs to $BV((a, b)^n, E)$ if the variation on lines

$$VL(f) \equiv \sum_{k=1}^n \int_{(a, b)^n} \text{ess} \, V_a^b f_{(k)}(y, \cdot) \, dy < \infty$$

and prove the following:

**Theorem 3.** (BV compactness) Suppose $E$ is a boundedly compact weakly separable metric space, $e_0 \in E$, and $0 < a < b < \infty$. Any sequence of functions $f_j \in L^1((a, b)^n, E)$ with

$$M \equiv \sup_j \int_{(a, b)^n} \text{dist}_E(f_j(x), e_0) \, dx + VL(f_j) < \infty$$

contains a subsequence $f_{j'}$ convergent pointwise a.e. and in $L^1$ to a function $f$ with

$$\int_{(a, b)^n} \text{dist}_E(f(x), e_0) \, dx + VL(f) \leq M.$$
Proof: We argue by induction on \( n \). The case \( n = 1 \) follows from Corollary 2. Assuming the theorem true for dimensions less than \( n \), we will use the metric space

\[
\tilde{E} \equiv \{ f \in L^1((a,b)^{n-1}, E) : \int_{(a,b)^{n-1}} \text{dist}_E(f(x), e_0) \, dx + VL(f) \leq M \} .
\]

One easily checks that \( \tilde{E} \) is weakly separable by again viewing \( E \) as a boundedly compact subset of \( \ell^\infty \). The inductive assumption guarantees that \( \tilde{E} \) is compact, hence boundedly compact.

For each \( k \in \{1, \ldots, n\} \), Fubini’s theorem implies that, for a.e. \( t \in \mathbb{R} \), each function \( f_{j(k)}(\cdot, t) \in \tilde{E} \) and that the map \( t \mapsto f_{j(k)}(\cdot, t) \) belongs to \( L^1((a,b), \tilde{E}) \) with the \( L^1 \) norm uniformly bounded by \( M \). Also for \( a \leq s < t \leq b \),

\[
\text{dist}_E(f_{j(k)}(\cdot, s), f_{j(k)}(\cdot, t)) = \int_{(a,b)^{n-1}} \text{dist}_E(f_{j(k)}(y, s), f_{j(k)}(y, t)) \, dy
\]

so that

\[
\text{ess} V_a^b f_{j(k)}(\cdot, \cdot) = \int_{(a,b)^{n-1}} \text{ess} V_a^b f_{j(k)}(y, \cdot) \, dy \leq M .
\]

The case \( n = 1 \) now gives \( L^1 \) convergence of a subsequence \( f_{j'}(\cdot, t) \) to a function \( g_k \in BV((a,b), \tilde{E}) \). We obtain \( g_1, g_2, \ldots, g_n \) by taking consecutive subsequences. The compatibility condition of the approximating functions

\[
f_{j(k)}(x'_k, x_k) = f_j(x_1, \ldots, x_n) = f_{j(l)}(x'_l, x_l)
\]

for \( k, l \in \{1, \ldots, n\} \) along with Lemma 1 implies the compatibility of these limit functions

\[
g_k(x'_k, x_k) = g_l(x'_l, x_l)
\]

which implies, using Fubini’s Theorem, the existence of a function well-defined by

\[
f(x_1, \ldots, x_n) = g_k(x'_k, x_k)
\]

for all \( k = 1, \ldots, n \) and almost all \( x \in (a,b)^n \). Also one has, by Fubini’s theorem the \( L^1 \) convergence

\[
\int_{(a,b)^{n-1}} \text{dist}_E(f_{j'}, f) \, dx = \int_a^b \text{dist}_E(f_{j'}(\cdot, t), f(\cdot, t)) \, dt \to 0
\]
as \( j \to \infty \). By Lemma 1, a subsequence of the \( f_{j'} \) also converges pointwise a.e. to \( f \). Measurability is a consequence of the pointwise convergence. One readily verifies the estimate

\[
\int_{(a,b)^{n}} \text{dist}_E(f(x), e_0) \, dx + VL(f) \leq M
\].