ANALYSIS QUALIFYING EXAM

August 2001

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. Suppose f is a positive bounded measurable function on $[0,1]$ and $F(x) = \int_0^x f(t)dt$ for $0 \le x \le 1$.
a. Show that F is continuous.
b. Show that F is differentiable at almost every point in $[0,1]$.

2. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, m and n are integers, and

$$(2 + |z|^m)^{-1} \frac{d^n f}{dz^n}$$

is bounded.
a. Prove that f is a polynomial.
b. Estimate the degree of f in terms of m and n.

3. Suppose f and g are integrable functions on $\mathbb{R}$. Show that the convolution $h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ is integrable with

$$\int_{-\infty}^{\infty} h(x) \;\le\; \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} g(x)dx \;.$$

4. Find $\int_0^\infty \frac{dx}{x^4+1}$.

5.
a. Show that any sequence f_n of positive integrable functions on $[0,1]$ with $\int_0^1 f_n^2\,dx \le \frac{1}{n^3}$ must converge to zero almost everywhere.
b. Is there a sequence g_n of positive integrable functions on $[0,1]$ satisfying $\int_0^1 g_n^2\,dx \to 0$ which does *not* converge to zero almost everywhere? Explain.

6. Suppose A is the annulus $\{z \in \mathbb{C} \,:\, 1 < |z| < 2\}$.
a. On A , is the function $\frac{1}{z}$ the *uniform* limit of a sequence of polynomials. Explain.
b. On A , is the function $\frac{1}{z-3}$ the *uniform* limit of a sequence of polynomials. Explain.