# ANALYSIS QUALIFYING EXAM

## AUGUST 2002

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

**1.** Suppose that $f$ is a nonnegative Lebesgue measurable function on $[0,1]$ and that $c = \int_0^1 f(x)\,dx < \infty$. Calculate:

$$\lim_{n\to\infty} \int_0^1 n \log(1 + \frac{f}{n})\,dx \ ,$$

and justify every step.

**2.** Let $A$ be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering $A$. Show that $\sum m(I_n) \geq 1$, where $m$ is the Lebesgue measure.

**3.** Suppose that $f$ is holomorphic and nonconstant in a bounded open, connected set $\Omega$ and that $\lim_{z\to z_0} |f(z)| = 1$ for all $z_0 \in \partial\Omega$.

a) Show that $f(\Omega)$ is a subset of the open unit disk.

b) Show that there is a point $w \in \Omega$ such that $f(w) = 0$.

c) Use part b) to show that $f(\Omega)$ is equal to the open unit disk. (*Hint:* Consider $\phi \circ f$ for suitable Möbius transformations $\phi$ that map the unit disk onto itself.

**4.** Suppose that $f(z)$ is holomorphic on the annulus $\{z \mid r < |z| < R\}$, and that the real part of $f$ is constant on each circle $\{z \mid |z| = \rho\}$ for $r < \rho < R$. Show that $f$ is a constant function. (*Hint:* consider $F(w) = f(e^w)$.)

**5.** (a) Suppose that $f$ is a meromorphic function on an open subset $\Omega$ of the complex plane and that

$$\int_\Omega |f(z)|^2\,dxdy \ < \ \infty.$$

Show that $f$ is actually *holomorphic* on all of $\Omega$.

(b) Give an example of a meromorphic function on $\{z \mid |z| < 1\}$ which is not holomorphic and which has

$$\int_{\{z \ | \ |z|<1\}} |f(z)|\,dxdy \ < \ \infty.$$

**6.** Assume that $f(x)$ is a Lebesgue measurable function on $\mathbb{R}$. Prove the function defined on $\mathbb{R}^2$ by

$$F(x,y) = f(x-y)$$

is Lebesgue measurable .