Please explain all your answers and indicate which theorems you are using.

1. Suppose $0 < \alpha < 2$.
   (a) Explain what is meant by the principal value integral $\int_0^\infty \frac{x^{\alpha}}{x-x^3} \, dx$.
   (b) Compute this integral.
   (c) Show that the answer you obtained in (b) is in agreement with the change of variable $x = \frac{1}{y}$ in the integral.

2. Suppose that $f(x, y)$ is continuous on the plane and that there is finite $M$ so that $|f(x, y) - f(x, z)| \leq M|y - z|$ for all $x, y, z \in \mathbb{R}$.
   (a) Show that, for any $x \in \mathbb{R}$, the partial derivative $\frac{\partial f}{\partial y}(x, y)$ exists for almost all $y \in \mathbb{R}$.
   (b) Prove that $\frac{d}{dy} \int_0^1 f(x, y) \, dx = \int_0^1 \frac{\partial f}{\partial y}(x, y) \, dx$.
   (c) Express $\frac{d}{dy} \int_0^{y^2} f(x, y) \, dx$ in terms of integrals of $f$ and $\frac{\partial f}{\partial y}$.

3. (a) Show that the direct analog of Rolle’s theorem does not apply to holomorphic functions. Do this by exhibiting an entire holomorphic function $f$ such that $f(0) = f(1)$ and yet $f'(z)$ never takes the value 0.
   (b) Suppose $f$ is a holomorphic function on the unit disk $\{ z : |z| < 1 \}$. Show that $f$ must be constant if $f(a_i) = f(b_i)$ for two sequences $a_i, b_i$ of positive real numbers that satisfy the inequalities
   $$0 < \ldots < a_{i+1} < b_{i+1} < a_i < b_i < \ldots < a_1 < b_1 < 1.$$

4. Suppose $0 < M < \infty$ and, for each positive integer $j$, $f_j : [0, 1] \to [-M, M]$ is a monotone increasing function. Prove that there is a subsequence $f_{j'}$ and a countable subset $A$ of $[0, 1]$ so that $f_{j'}(t)$ converges, as $j' \to \infty$, for every $t \in [0, 1] \setminus A$.

5. (a) Is there a nonconstant real function $h$ that is continuous on the closed disk $\{ z : |z| \leq 1 \}$, harmonic on the open disk $\{ z : |z| < 1 \}$, and vanishes on the upper unit semi-circle (that is, $h(e^{i\theta}) = 0$ for $0 \leq \theta \leq \pi$)?
   (b) Is there a nonconstant complex function $f$ that is continuous on the closed disk $\{ z : |z| \leq 1 \}$, holomorphic on the open disk $\{ z : |z| < 1 \}$, and vanishes on the upper unit semi-circle (that is, $f(e^{i\theta}) = 0$ for $0 \leq \theta \leq \pi$)?

6. Assume that $f(x)$ is a Lebesgue measurable function on $\mathbb{R}$. Prove the function defined on $\mathbb{R}^2$ by $F(x, y) = f(x - y)$ is Lebesgue measurable.