Analysis Exam, Fall 2005

1. (a) Suppose $f : [0,1] \to \mathbf{R}$ is Lebesgue integrable. Find

$$\lim_{n\to\infty} \int_0^1 \frac{nx^2}{2+nx}\, f(x)\, dx \ ,$$

and justify each of your steps.

(b) Give an example of a Lebesgue integrable $f : [0,1] \to \mathbf{R}$ for which

$$\lim_{n\to\infty} \int_0^1 \frac{nx}{2+nx^2}\, f(x)\, dx \ = \ +\infty \ .$$

2. Suppose g is holomorphic on $\{z \in \mathbf{C} \ : \ |z| < 2\}$ and $|g(z)| < 1$ whenever $|z| = 1$. Show that there exists a unique point $w \in \mathbf{C}$ with $|w| < 1$ and $g(w) = w$.

3. Suppose $f : \mathbf{R} \to \mathbf{R}$ is infinitely differentiable.

(a) Define carefully, for each point a with $f(a) = 0$, the *order of vanishing* $N(f,a)$ of f at a. Hint: $N(f,a) \in \{1,2,\ldots\} \cup \{+\infty\}$.

(b) Let $N(f) = \sum_{\{a\,:\,f(a)=0\}} N(f,a)$. (Here $N(f) = +\infty$ in case f has infinitely many zeros.) Prove the inequality:

$$N(f') \ \geq \ N(f) \ - \ 1 \ .$$

4. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x - \mathbf{i}}\, dx \quad .$$

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is infinitely differentiable.

(a) Derive a formula for

$$\int_0^x f''(t)(x-t)\, dt$$

in terms of $f(x),\ f(0),\ f'(0)$.

(b) Give an upper bound for $|f(x)|$ in terms of

$$|f(0)|,\ |f'(0)|, \ \text{ and } \ M = \sup_{|t|\leq|x|} |f''(t)| \ .$$

6. Suppose g is holomorphic on $A = \{z \in \mathbf{C} \ : \ 0 < |z| < 1\}$ and

$$\limsup_{|z|\to 0} |g(z) - \lambda| \ > \ 0$$

for every $\lambda \in \mathbf{C}$. Show that either

(I) $\lim_{|z|\to 0} |z|^{1/2}|g(z)| \ = \ \infty$ or

(II) $g(A)$ is dense in $\mathbf{C}$.

Give a specific example of a g satisfying (II).