1. (a) Suppose \( f : [0,1] \to \mathbb{R} \) is Lebesgue integrable. Find
\[
\lim_{n \to \infty} \int_0^1 \frac{nx^2}{2 + nx} f(x) \, dx .
\]
Noting the bound,
\[
\left| \frac{nx^2}{2 + nx} f(x) \right| \leq |x||f(x)| \leq |f(x)| ,
\]
we use Lebesgue’s Dominated Convergence Theorem to see that the limit is \( \int_0^1 x f(x) \, dx \).

(b) Taking \( f \equiv 1 \) and noting that \( nx^2 + nx^2 \) is increasing in \( n \) and approaches \( 1/x \) as \( n \) approaches \( \infty \), we conclude from the Monotone Convergence Theorem or Fatou’s Lemma that
\[
\lim_{n \to \infty} \int_0^1 \frac{nx}{2 + nx^2} \, dx = +\infty .
\]
Alternately one can substitute to compute that
\[
\int_0^1 \frac{nx}{2 + nx^2} \, dx = \frac{1}{2} \log(1 + \frac{n}{2}) \to +\infty \text{ as } n \to \infty .
\]

2. Suppose \( g \) is holomorphic on \( \{ z \in \mathbb{C} : |z| < 2 \} \) and \( |g(z)| < 1 \) whenever \( |z| = 1 \).
The function \( h(z) = z - g(z) \) is holomorphic on \( \{ z \in \mathbb{C} : |z| < 2 \} \) and satisfies
\[
|h(z)| = |g(z)| < 1 = |z|
\]
on the unit circle. By Rouché’s Theorem \( h \) has, like the function \( z \), a single simple zero \( w \) in the unit disk. This is the desired unique point \( w \in \mathbb{C} \) with \( |w| < 1 \) and \( g(w) = w \).

3. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is infinitely differentiable and \( a \in \mathbb{R} \).

(a) The order of vanishing
\[
N(f,a) = \sup \{ n : f^{(n)}(a) = 0 \} .
\]
Thus \( N(f,a) \) is a positive integer in case some derivative of \( f \) at \( a \) is nonzero, and \( N(f,a) = \infty \) in case the derivatives of \( f \) at \( a \) of all orders vanish.

(b) For convenience, we also define \( N(f,b) = 0 \) for a point \( b \) with \( f(b) \neq 0 \). Then for any zero \( a \) of \( f \), we immediately verify the relation \( N(f',a) = N(f,a) - 1 \). Also the Mean Value Theorem implies that, strictly between any 2 consecutive zeros of \( f \), is a zero
of \( f' \). It then follows that \( N(f') = \infty \) whenever \( N(f) = \infty \). So we now may assume that \( N(f) < \infty \). We then find that
\[
N(f') = \sum_{a \in (f')^{-1}\{0\}} N(f', a) \\
\geq \# [(f')^{-1}\{0\} \setminus f^{-1}\{0\}] + \sum_{a \in f^{-1}\{0\}} N(f', a) \\
\geq [\#f^{-1}\{0\} - 1] + \sum_{a \in f^{-1}\{0\}} [N(f, a) - 1] = N(f) - 1.
\]

4. Evaluate the integral
\[
\int_{-\infty}^{\infty} \frac{\cos x}{x - i} \, dx.
\]
For \( R \) large, consider the upper disk bounded by the interval \([-R, R]\) and the semicircle \( \Gamma_R = \{Re^{it} : 0 \leq t \leq \pi\} \).

One is tempted to apply the Residue Theorem with the meromorphic function \( \frac{\cos z}{z - i} \).
However the upper boundary integral does not approach 0 as \( R \to \infty \). Note that \( \cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \) and that
\[
|\int_{\Gamma_R} \frac{e^{iz}}{z - i} \, dz| \leq \int_0^{2\pi} \left| \frac{e^{iz}Re^{it}}{Re^{it} - i} \right| |Re^{it}| \, dt \\
\leq \frac{2R}{R} \int_0^{2\pi} e^{-R\sin t} \, dt \to 0 \text{ as } R \to \infty,
\]
by Dominated Convergence. Since the function \( \frac{e^{iz}}{z - i} \) has only one pole at \( i \) with residue \( e^{-1} \), the Residue Theorem gives that
\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x - i} \, dx = 2\pi ie^{-1}.
\]
To find the other term \( \int_{-\infty}^{\infty} \frac{e^{-ix}}{x - i} \, dx \), we use the lower half disk with lower boundary \( \gamma_R = \{-Re^{it} : 0 \leq t \leq \pi\} \) and see that, just like above,
\[
|\int_{\gamma_R} \frac{e^{-ix}}{z - i} \, dz| \to \infty \text{ as } R \to \infty.
\]
The meromorphic function \( \frac{e^{-ix}}{z - i} \) has no pole in the lower half plane. So the Residue Theorem now gives
\[
\int_{-\infty}^{\infty} \frac{e^{-ix}}{x - i} \, dx = 0,
\]
and we conclude that \( \int_{-\infty}^{\infty} \frac{\cos z}{z - i} \, dx = \pi ie^{-1} \).
5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable.

(a) $\int_0^x f''(t)(x-t) \, dt$ is the integral form for the remainder term in a degree 1 Taylor approximation of $f$ and so equals $f(x) - f(0) - f'(0)x$. One can also prove this directly by integrating by parts:

\[
\int_0^x f''(t)(x-t) \, dt = f'(x)(x-x) - f'(0)(x-0) - \int_0^x f'(t) \frac{d}{dt}(x-t) \, dt = -f'(0)x + \int_0^x f'(t) \, dt = -f'(0)x + f(x) - f(0).
\]

(b) From (a)

\[
|f(x)| \leq |f(0) + f'(0)x + \int_0^x f''(t)(x-t) \, dt| \leq |f(0)| + |f'(0)||x| + M \int_0^x |x-t| \, dt = |f(0)| + |f'(0)||x| + \frac{1}{2}M|x|^2
\]

for $x > 0$. For $x < 0$ one gets the same estimate by symmetry.

6. Suppose $g$ is holomorphic on $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and

\[
\limsup_{|z| \to 0} |g(z) - \lambda| > 0
\]

for every $\lambda \in \mathbb{C}$. Show that either

(I) $\lim_{z \to 0} |z|^{1/2}|g(z)| = \infty$ or

(II) $g(A)$ is dense in $\mathbb{C}$.

The hypothesis implies that the singularity of $f$ is not removable and thus is either (I) a pole or (II) an essential singularity. In case (I), $g$ is meromorphic at 0 and so, near 0, $|g(z)| \geq \frac{c}{|z|^j}$ for some $c > 0$ and $j \in \{1, 2, \ldots\}$, hence,

\[
\lim_{|z| \to 0} |z|^{1/2}|g(z)| \geq \frac{c}{|z|^{j-1/2}} = \infty.
\]

In any neighborhood of an essential singularity, $g$ becomes arbitrarily close to any complex number so that $g(A)$ is dense in case (II).

A specific example of a $g$ satisfying (II) is $g(z) = e^{1/z}$.