ANALYSIS QUALIFYING EXAM
September 1998

Justify answers as completely as you can. Give careful statements of theorems you are using. Time limit – 3 HOURS.

1. (a) For what real numbers $\alpha$ is $|x|^\alpha$ integrable on $\{x \in \mathbb{R}^n : |x| < 1\}$?
(b) For what real numbers $\beta$ is $|x|^\beta$ integrable on $\{x \in \mathbb{R}^n : |x| > 1\}$?

2. (a) How many roots does $p(z) = 2z^5 + 4z^2 + 1$ have in the disk $|z| < 1$?
(b) How many roots does the same polynomial have on the real axis?

3. Let $P_n(x) = a_n x^2 + b_n x + c_n$ be a sequence of quadratic polynomials which converges pointwise on $0 \leq x \leq 1$.
(a) Prove that $P_n$ converges uniformly on $[0,1]$.
(b) Does $P_n$ converge uniformly on $[0,2]$?

4. Evaluate the improper integral
$$\int_{-\infty}^{\infty} \left( \frac{\sin x}{x} \right)^2 \, dx.$$ [Hint: Express $\sin^2 x$ in terms of $e^{2ix}$.]

5. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$, $0 < f(0)$ and $f(100) < 100$.
(a) Prove that $f(x) = x$ for some $x$ if $f$ is continuous.
(b) Prove that $f(x) = x$ for some $x$ if $f$ is monotonically increasing (though possibly discontinuous).

6. What is the general form of an entire function which has absolute value 1 on the circle $|z| = 1$ and has no zero inside the circle? Prove your result.