ANALYSIS QUALIFYING EXAM
AUGUST 1999

Justify answers as completely as you can. Give careful statements of theorems you are using. Time limit – 3 HOURS.

1. Suppose that \( f_1, f_2, \ldots \) are nonnegative continuous functions on \([0, 1]\) with \( \int_0^1 f_n(x) \, dx \leq M \).
   (1) Show that there exists a point \( a \in [0, 1] \) with \( f_1(a) \leq 2M \) and \( f_2(a) \leq 2M \).
   (2) Does there exist a better estimate? That is, a number \( N < M \) so that \( \inf_{0 \leq a \leq 1} \max\{f_1(a), f_2(a)\} \leq N \) for all such \( f_1, f_2 \). If so, find the smallest such \( N \). If not, give a counterexample.
   (3) Show that there always exists an \( a \in [0, 1] \) so that \( f_n(a) \leq M \) for infinitely many \( n \).

2. Suppose \( f \) is a holomorphic function on \( \{ z \mid |z| < 3R \} \), \( f(0) = 0 \), \( M_R = \sup_{|z| \leq R} |f(z)| \), and \( N_R = \sup_{|z| \leq R} |f'(z)| \).
   (1) Estimate \( M_R \) (from above) in terms of \( N_R \).
   (2) Estimate \( N_R \) (from above) in terms of \( M_{2R} \).

3. Suppose that \( f(x) \) is defined on \([-1, 1]\), and that \( f'''(x) \) is continuous. Show that the series
   \[ \sum_{n=1}^{\infty} (n(f(1/n) - f(-1/n)) - 2f'(0)) \]
   converges.

4. Prove that there is no one-to-one conformal map of the punctured disc \( G = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \) onto the annulus \( A = \{ z \in \mathbb{C} \mid 1 < |z| < 2 \} \).

5. Let \( f \) is a meromorphic function on the complex plane such that \( f(z) = 1 + z + z^2 + \cdots \) whenever \( |z| < 1 \). Define a sequence of real numbers \( a_0, a_1, a_2, \ldots \) by
   \[ f(z) = \sum_{n=0}^{\infty} a_n (z + 2)^n \]
   What is the radius of convergence of the new series \( \sum_{n=0}^{\infty} a_n z^n \)?

6. A function \( g : [0, 1] \to \mathbb{R} \) is concave if \( tg(x) + (1 - t)g(y) \leq g(tx + (1 - t)y) \) for \( 0 \leq t \leq 1 \). Prove that for any continuous \( f : [0, 1] \to \mathbb{R} \) with \( f(0) = 0 \), there is a continuous concave function \( g : [0, 1] \to \mathbb{R} \) such that \( g(0) = 0 \) and \( g(x) \geq f(x) \) for all \( x \in [0, 1] \).
   (Hint: Show that \( g(x) = \inf \{ h(x) : h \text{ is a continuous concave function on } [0, 1], h(y) \geq f(y) \text{ for } y \in [0, 1] \} \))
works (in particular, \( g(0) = 0 \)).