ANALYSIS QUALIFYING EXAM

January 2000

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. Do there exist function $f(z)$ that is analytic at $z = 0$ and that satisfy

$$f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3}, \qquad n = 1,\ 2, \ldots$$

2. Let $\{f_n\}$ be a sequence of continuous maps $[0,1] \to \mathbb{R}$ such that

$$\int_0^1 (f_n(y))^2 dy \le 5$$

for all n. Define $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x) = \int_0^1 \sqrt{x+y} f_n(y) dy.$$

Prove that a subsequence of the sequence $\{g_n\}$ converges uniformly.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

Show that there is a sequence $x_n \in \mathbb{R}$ such that $x_n \to \infty$, $x_n f(x_n) \to 0$ and $x_n f(-x_n) \to 0$ as $n \to \infty$.

4. Let f be a holomorphic map of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ into itself which is not the identity map $f(z) = z$. Show that f can have at most one fixed point.

5. Let $g(z)$ be analytic in the right half-plane $Rez > 0$, with $|g(z)| < 1$ for all such z. If $g(1) = 0$ how large can $|g(2)|$ be ?

6. Let f be a C^2 function on the real line. Assume f is bounded with bounded second derivative. Let

$$A = sup_{x \in \mathbb{R}} |f(x)|, \qquad B = sup_{x \in \mathbb{R}} |f''(x)|.$$

Prove that

$$sup_{x \in \mathbb{R}} |f'(x)| \le 2\sqrt{AB}.$$