ANALYSIS QUALIFYING EXAM

January 2000

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. Do there exist function $f(z)$ that is analytic at $z = 0$ and that satisfy

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}, \quad n = 1, 2, \ldots$$

2. Let $\{f_n\}$ be a sequence of continuous maps $[0, 1] \rightarrow \mathbb{R}$ such that

$$\int_0^1 (f_n(y))^2 dy \leq 5$$

for all $n$. Define $g_n : [0, 1] \rightarrow \mathbb{R}$ by

$$g_n(x) = \int_0^1 \sqrt{x + y} f_n(y) dy.$$  

Prove that a subsequence of the sequence $\{g_n\}$ converges uniformly.

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, with

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$  

Show that there is a sequence $x_n \in \mathbb{R}$ such that $x_n \rightarrow \infty$, $x_n f(x_n) \rightarrow 0$ and $x_n f(-x_n) \rightarrow 0$ as $n \rightarrow \infty$.

4. Let $f$ be a holomorphic map of the unit disk $D = \{ z : |z| < 1 \}$ into itself which is not the identity map $f(z) = z$. Show that $f$ can have at most one fixed point.

5. Let $g(z)$ be analytic in the right half-plane $Rez > 0$, with $|g(z)| < 1$ for all such $z$. If $g(1) = 0$ how large can $|g(2)|$ be?

6. Let $f$ be a $C^2$ function on the real line. Assume $f$ is bounded with bounded second derivative. Let

$$A = \sup_{x \in \mathbb{R}} |f(x)|, \quad B = \sup_{x \in \mathbb{R}} |f''(x)|.$$  

Prove that

$$\sup_{x \in \mathbb{R}} |f'(x)| \leq 2\sqrt{AB}.$$