ANALYSIS QUALIFYING EXAM

January 2000

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. Suppose $g(x) = \lim_{n\to\infty} g_n(x)$ for $x \in [0,1]$ where each g_n is a positive continuous function on $[0,1]$ with $\int_0^1 g_n\,dx = 1$.

a. Is it always true that $\int_0^1 g(x)\,dx \leq 1$? Prove or find a counter-example.

b. Is it always true that $\int_0^1 g(x)\,dx \geq 1$? Prove or find a counter-example.

2. Let f be a complex-valued function in the open unit disc,$\mathbb{D}$, of the complex plane such that the functions $g = f^2$ and $h = f^3$ are both analytic. Prove that f is analytic in $\mathbb{D}$.

3. Construct an open set $U \subset [0,1]$ such that U is dense in $[0,1]$, the Lebesgue measure $\mu(U) < 1$, and that $\mu(U \cap (a,b)) > 0$ for any interval $(a,b) \subset [0,1]$.

4. Let f, g_1, g_2, $\cdots$ be entire functions. Assume that the kth derivatives at 0 satisfy

a. $|g_n^{(k)}(0)| \leq |f^{(k)}(0)|$ for all n and k;

b. $\lim_{n\to\infty} g_n^{(k)}(0)$ exists for all k. Prove that the sequence $\{g_n\}$ converges uniformly on compact sets and that its limit is an entire function.

5. Let f be a continuous, strictly increasing function from $[0,\infty)$ onto $[0,\infty)$ and let $g = f^{-1}$ be the inverse function of f . Prove that

$$\int_0^a f(x)dx + \int_0^b g(x)dx \geq ab$$

for all positive numbers a and b.

6. Suppose that $f(z)$ that is analytic and satisfies $f(\frac{1}{z}) = f(z)$ for all $z \in \mathbb{C}\backslash\{0\}$.

a. Write down the general *Laurent expansion* for f.

b. Show that the coefficients of this expansion are all real if this f has real values on the unit circle $|z| = 1$.