ANALYSIS QUALIFYING EXAM

January 2004

Please explain *all* your answers and indicate which theorems you are using.

1. (a) Classify all entire functions $f : \mathbf{C} \to \mathbf{C}$ such that

$$\limsup_{R\to\infty} \sup_{|z|=R} \frac{|f(z)|}{R^4} < \infty .$$

(b) Classify all entire functions $f : \mathbf{C} \to \mathbf{C}$ such that

$$\liminf_{R\to\infty} \inf_{|z|=R} \frac{|f(z)|}{R^4} > 0 .$$

2. Suppose that $f_n : \mathbf{R} \to \mathbf{R}$ is a differentiable function for every positive integer n, $M = \sup_{n,x} |f_n'(x)| < \infty$ and that $f(x) = \lim_{n\to\infty} f_n(x) \in \mathbf{R}$ exists for all $x \in \mathbf{R}$.

(a) Show that the functions f_n are *uniformly bounded* on each fixed interval $[-R, R]$.

(b) Is f *continuous* on $\mathbf{R}$? Prove or find a counterexample.

(b) Is f *differentiable* on $\mathbf{R}$? Prove or find a counterexample.

3. Compute the (improper) integral

$$\int_0^\infty \frac{\sin x}{x(x^2+1)(x^2+2)^2}\, dx .$$

4. (a) In the unit disk $\{z \in \mathbf{C} : |z| < 1\}$ how many solutions are there to the equation $z^8 - 5z^3 + z = 2$?

(b) In the radius-2 disk $\{z \in \mathbf{C} : |z| < 2\}$ how many solutions are there to the same equation $z^8 - 5z^3 + z = 2$?

5. (a) Suppose that f is integrable on $[0, 1]$. Show that there exists a decreasing sequence $a_n \downarrow 0$ so that $\lim_{n\to\infty} a_n |f(a_n)| = 0$.

(b) Let f_n be a sequence of functions integrable on $[0, 1]$ with $\sup_n \int_0^1 |f_n(x)|\, dx < \infty$. Does there exist a subsequence f_{n_k} of f_n and sequence of points $b_k \downarrow 0$ and so that $\lim_{k\to\infty} b_k |f_{n_k}(b_k)| = 0$. If so, prove it. If not, find a counterexample.

6. Suppose $1 \le p \le \infty$, $f \in L^p\big([0, 1]\big)$, and $g(t)$ is the Lebesgue measure of the set $\{x \in [0, 1] : |f(x)| > t\}$ for $0 \le t < \infty$.

(a) Show that $\int_0^\infty g(t)\, dt < \infty$ if $1 < p \le \infty$.

(b) Is this still true for $p = 1$? Prove or find a counterexample.