ANALYSIS QUALIFYING EXAM
January 2004

Please explain all your answers and indicate which theorems you are using.

1. (a) Classify all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that
   \[
   \limsup_{R \to \infty} \sup_{|z| = R} \frac{|f(z)|}{R^4} < \infty.
   \]
   (b) Classify all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that
   \[
   \liminf_{R \to \infty} \inf_{|z| = R} \frac{|f(z)|}{R^4} > 0.
   \]

2. Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a differentiable function for every positive integer $n$,
   $M = \sup_{n,x} |f'_n(x)| < \infty$ and that $f(x) = \lim_{n \to \infty} f_n(x) \in \mathbb{R}$ exists for all $x \in \mathbb{R}$.
   (a) Show that the functions $f_n$ are uniformly bounded on each fixed interval $[-R, R]$.
   (b) Is $f$ continuous on $\mathbb{R}$? Prove or find a counterexample.
   (b) Is $f$ differentiable on $\mathbb{R}$? Prove or find a counterexample.

3. Compute the (improper) integral
   \[
   \int_0^\infty \frac{\sin x}{x(x^2 + 1)(x^2 + 2)^2} \, dx.
   \]

4. (a) In the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ how many solutions are there to the equation
   $z^8 - 5z^3 + z = 2$?
   (b) In the radius-2 disk $\{z \in \mathbb{C} : |z| < 2\}$ how many solutions are there to the same equation $z^8 - 5z^3 + z = 2$?

5. (a) Suppose that $f$ is integrable on $[0, 1]$. Show that there exists a decreasing sequence
   $a_n \downarrow 0$ so that $\lim_{n \to \infty} a_n |f(a_n)| = 0$.
   (b) Let $f_n$ be a sequence of functions integrable on $[0, 1]$ with $\sup_n \int_0^1 |f_n(x)| \, dx < \infty$.
   Does there exist a subsequence $f_{n_k}$ of $f_n$ and sequence of points $b_k \downarrow 0$ and so that
   $\lim_{k \to \infty} b_k |f_{n_k}(b_k)| = 0$. If so, prove it. If not, find a counterexample.

6. Suppose $1 \leq p \leq \infty$, $f \in L^p([0, 1])$, and $g(t)$ is the Lebesgue measure of the set
   $\{x \in [0, 1] : |f(x)| > t\}$ for $0 \leq t < \infty$.
   (a) Show that $\int_0^\infty g(t) \, dt < \infty$ if $1 < p \leq \infty$.
   (b) Is this still true for $p = 1$? Prove or find a counterexample.